INVERSE PROBLEMS IN EXPERIMENTAL SOLID MECHANICS

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Contents

Summary

This chapter browses the use of inverse procedures in experimental solid mechanics. After a general introduction on inverse problems including a description of the different fields of application in general solid mechanics, the topic is narrowed down to the important area of the identification of the mechanical properties of materials, with a specific view to the use of full-field measurements. For this particular case, a more

dedicated definition of the inverse problem to solve is given and the main routes to solve it are described. Finally, examples of experimental applications of such procedures are presented.

1. General Introduction to Inverse Problems in Solid Mechanics

Classically in solid mechanics, experimentalists put most of their efforts in characterizing the mechanical behavior of materials and structures by a set of constants or curves whereas the theoreticians put most of their efforts in developing theories and models that are consistent with the curves and the constants found by the experimentalists.

Due to the exponential increase in computational power and the continuous development of novel and more complex experimental techniques, we are experiencing a very unique time in the history of solid mechanics, in which models and experiments can be compared directly. This opens new prospects for characterizing more precisely the mechanical behavior of complex systems and materials. Indeed, it is possible to deduce directly from the experiments the parameters of the models developed by the theoreticians. However, this entails solving what is called an "*inverse problem*".

The objective of the first section of this paper is to give an overview of the general issues related to *inverse problems* in experimental solid mechanics. Since the subject is very broad, this section will contain very general information together with a number of key references so that interested readers can easily set foot in the subject.

The rest of this contribution will be dedicated to one of the major source of application of *inverse problems* in experimental solid mechanics: the identification of the parameters of mechanical constitutive behavior laws of materials, with a specific view to the use of full-field measurements.

1.1. Significance and Importance

The theories of mechanics and dynamics of deformable bodies allow us to make predictions: given a complete description of a system, we can predict the outcome of some measurements. This problem of predicting the result of measurements is called the *forward problem*. The *inverse problem* consists in using the actual result of some measurements to infer the values of the parameters that characterize the system.

The main types of inverse problems that arise in the mechanics of deformable solids are similar to those encountered in other areas of physics involving continuous media and distributed physical quantities, e.g., acoustics, electrostatics and electromagnetism. They are usually motivated by the desire or need to overcome a lack of information concerning the properties of the system (a deformable solid body or structure). This desire or need exists in several domains of mechanics, with many industrial or health applications:

- In soil mechanics, e.g. for detecting underground oil or other raw materials,
- In seismology, e.g. for modeling earthquakes,

- In aerospace and aeronautics, e.g. for updating dynamic models of large structures,
- In mechanics of materials, e.g. for determining the mechanical properties of materials,
- In medicine, e.g. for detecting tumors,
- In manufacturing, for detecting defects through non destructive testing…

A forward problem is generally formulated as follows: if p is a set of model parameters and u is a set of observables (data), there exists an operator, generally nonlinear, denoted g, which gives the error free values of u such as:

$$u = g(p) \tag{1}$$

Operator g is determined by solving the classical partial differential equations (PDEs) governing the mechanical behavior of solids or systems of solids. These equations may be written like this:

$$M(u, p) = 0 \tag{2}$$

where M is an operator featuring the equations of the forward problem (equilibrium equations, constitutive equations and compatibility equations), which involve the observables and the unknown parameters. For example, in quasi-static analysis, M takes generally the form:

$$M(u, p) = K(p).u - F \tag{3}$$

where K is the stiffness operator, the observables u are local displacement values and F are local force values.

The inverse problem consists in determining the operator giving the set of model parameters p that corresponds to the measured data set u, in other words the set of model parameters p that are in agreement with both the measured data set u and operator M in Eq. (2). An objective function that minimizes the differences or misfit between the observed states and those predicted by the g operator through PDEs is formulated, and solution of a nonlinear optimization problem yields components of the unknown parameters.

The inverse problem is often significantly more difficult to solve than the forward problem, because:

- the forward solution is just a sub-problem of the inverse problem;
- the inverse problem is often ill-posed despite the well-posedness of the forward problem (a well-posed problem in mathematics is a problem for which some theorems prove the existence and uniqueness of the solution);
- the inverse operator couples the entire time history of the system’s response, as opposed to the usual case with forward evolution operators;
- the inverse problem can have numerous local solutions.

All these difficulties induce numerical issues in the minimization of the objective functions. Depending on the scale of the problem to solve and on its degree of ill-posedness, different approaches may be used: evolutionary algorithms, Newton's method, quasi-Newton methods (like L-BFGS), steepest descent… Minimization is always constrained: one has to minimize the differences between measured and predicted data, under the constraints that the predicted ones satisfy the PDEs that yield Eq. (1). It is said that optimization is PDE-constrained. Inequality constraints may also be added for bounding the values of the unknown parameters.

A large number of inverse problems are defined using a least-squares PDE-constrained objective function, which is minimized by a gradient-based approach (Newton or quasi-Newton). Accordingly, the equations that determine the solution can be derived by requiring stationarity of a Lagrangian functional. In general, these so-called optimality conditions take the form of a coupled nonlinear three-field system of integro-partial-differential equations in the state variable u, the adjoint variable λ, and the model parameters p.

The problem as formulated is ill-posed: the solution may neither be unique nor depend continuously on the given data. Discretization of the inverse problem leads to inverse operators that are ill-conditioned and rank deficient. Ill-posedness may occur for several reasons:

1. the predicted values cannot, in general, be identical to the measured values for two reasons: measurement uncertainties and model imperfections. For this reason, it is generally not possible to set inverse problems properly without a careful analysis of model and measurement uncertainties;
2. the operator g is not systematically bijective, which means that the same values of u can be predicted with two different sets of model parameters, or that some values of u cannot be predicted with any set of model parameters.

These difficulties may be overcome by different means:

1. one remedy is to regularize by penalizing very improbable solutions. Improbable solutions can only be discarded when some *a priori* information is known about the model. All the success of this procedure will lie in the amount and relevancy of this *a priori* knowledge. For example, when the unknown model parameters represent the spatial distribution of some mechanical property that is known to vary gradually, it may be useful to regularize with the L_2 norm of the gradient of this distribution. Such Tikhonov regularization eliminates oscillatory components of p from being possible solutions. Some authors have shown that Tikhonov regularization may smooth possible discontinuities in a model and promote the use of total variation regularization functional instead;

2. another remedy is to give the solution of the inverse problem in terms of statistics. In the space of possible parameters p, each set of parameters is characterized by a probability. This probability may be computed from probability densities

representing possible *a priori* information and the uncertainties. From the obtained probability, different solutions may be defined: the mean solution, the median solution, the maximum likelihood solution. Some authors also give the solution in terms of intervals;

3. a practical remedy is also to refine the experiments. Indeed, the sensitivity of the cost function to a given unknown parameter can be completely flattened as a result of inappropriate density and location of sources (or loading points) and receivers (or measurement points). Improving these densities and/or locations can be achieved by investigating and optimizing parameter sensitivities. Changing the experimental techniques can also help to reduce ill-posedness of some inverse problems. However, it may also have the reverse effect because, as soon as new experimental methods are capable of decreasing the experimental uncertainty or increasing the measurement density, new theories and new models with a larger number of unknown parameters arise that allow to account for the observations more accurately.

Nowadays, there are two major prospects for the solid mechanics scientists working on inverse problems:

- the first one is experimental: it lies in the reduction of the solution uncertainty of classical inverse problems that have been existing for a long time (development of new experimental techniques, multiplication of experimental tests, design of new experiments, model modifications…);
- the second is mathematical and numerical: it lies in the numerical issues arising from the resolution of novel inverse problems of very large scales (earthquake modeling, molecular modeling, systems with multiphysical couplings…).

1.2. Main Types of Practical Inverse Problems

Inverse problem may be sorted in function of the type of parameter that is sought. According to this criterion, inverse problems in mechanics of deformable solids may be sorted in seven groups:

1. the reconstruction of buried objects of a geometrical nature, such as cracks, cavities or inclusions. Crack identification problems consist in identifying a crack (or a set of cracks) from a set of over-determined force–displacement boundary measurements. Such problems can be formulated within different physical contexts such as electrostatics, elasto-magnetism, acoustics or elasto-dynamics. They have important applications in, for example, non-destructive testing or identification of seismic faults. Many strategies have been proposed for solving crack identification problems. In particular, reciprocity gap functionals can be formulated and applied to crack identification problems;

2. the non-destructive testing (NDT) of solids or structures using mechanical waves, such as ultrasonic, Rayleigh or Lamb waves. NDT with waves has been widely applied to engineering infrastructures and archaeological objects. Depending on the size of the objects under test and the spatial resolution requirements, waves with different frequencies can be used. The key is to identify the stable and reliable

dispersion features of the entire frequency band, especially those of high frequencies. This may be achieved through the precise analysis of phase differences between waves of different frequencies on two measuring points.

3. the identification of distributed parameters (e.g., elastic moduli, mass density, wave velocity), having applications for example, in medical imaging of tissues where the objective is to characterize the stiffness of tissues in order to detect diseases or tumors or in seismic exploration and geomechanics where the objective is to reconstruct the distributed constitutive parameters of the different underground strata. Many applications have been addressed by combining external loadings and external measurements: displacements induced by an applied static or dynamic loading are measured over the external surface of the solid and linked to the internal properties. In medical sciences for instance, this technique has been applied "manually" for many years by physicians trying to detect hard tissues by external palpations. Even if techniques combining external loadings and external measurements have been successful in many applications, it is now possible to combine external loading and internal (bulk) tridimensional (3D) full-field measurements. In medical sciences, thanks to ultrasound and magnetic resonance imaging (MRI), it is possible to measure displacement fields in many human tissues. In the domain of mechanics of materials, experimental techniques such as X-ray or neutron diffraction tomography, or wavelength scanning interferometry are becoming available for bulk measurements in certain materials.

4. Efficient and powerful approaches have still to be developed for processing this type of full-field data within the final objective of constitutive parameter identification. A lot of work has already been achieved in medical sciences, where full-field measurements have been available for more than a decade. The measurement of tissue elasticity has even become an independent branch, called "elastography". The approaches used to process the full-field data depend on the way the solid is deformed: static experiments where the tissue is compressed quasi-statically, and dynamic experiments: a time harmonic excitation on the boundary creates a time harmonic shear wave in the tissue;

5. the reconstruction of residual stresses, which has important engineering implications. Many techniques have been developed to measure internal stresses. Embedded sensors such as optical fibers and strain gages can be to measure the local internal stress state. For laminated composite materials for instance, at a ply or macroscopic scale, the hole-drilling method, the layer removal method and the compliance method may be used. The compliance method applied to filament wound tubes consists in sectioning the tubes and measuring the change in strains on the external and internal surfaces with two biaxial strain gages, one bonded to each side of the tube wall. This leads to the estimation of the axial and circumferential internal bending moments of filament wound tubes. Using the hole-drilling or other removal approaches in combination with full displacement field optical techniques may provide some advantages with respect to the traditional use of strain gauges, in particular higher sensitivity and no-contact measurement.

6. the correction of local parameters that are not known with sufficient accuracy in the

models of complex engineering structures.

7. the identification of sources or inaccessible boundary values (i.e., Cauchy problems in elasticity) are also encountered.

8. eventually, the identification of constitutive properties in solids or structures for which simplifying assumptions such as constant states of strain or stress are invalid. The literature about the six former points is dense and several articles surveying the subject can be found in the literature. This latter point, regarding the identification of constitutive properties in solids and structures (homogeneous or slowly varying properties), has seen dramatic improvements during the last decade thanks to the development of full-field measurements. Such experimental techniques yield rich experimental data and are therefore well suited to identification problems. The importance of the latter techniques is increasing as inversion techniques specifically exploiting availability of field quantities (either on the boundary or over part of the domain itself) become accessible. Due to the importance of these techniques and also due to their recent development, the rest of the article will be fully dedicated to these inversion techniques.

2. Identification of Constitutive Behavior of Materials: Theory

2.1. Statement of the Problem

The practical identification of the parameters governing the constitutive equations of structural materials is a key issue in experimental solid mechanics. It usually relies on performing some simple mechanical tests such as tensile or bending tests. In these cases, a closed form solution for the corresponding mechanical problem provides a direct link between unknown parameters and measurements which are usually local strain components and applied loads. This approach, called “direct inversion”, is presented briefly in Section 2.2.1. However, such a classical approach suffers from two main drawbacks which can be summarized as follows:

- the assumptions needed to link directly the unknown parameters to the load and strains are usually rather stringent (uniform pressure distribution at the specimens ends for the simple tensile test, for instance). These requirements are particularly difficult to meet for anisotropic materials, for instance. When heterogeneous materials are tested (welds, functionally graded materials...), it is even more critical;

- only a small number of parameters can be determined with such mechanical tests, hence the need to perform several tests when the constitutive equations depend on more parameters than for isotropy (anisotropic elasticity, plasticity for instance).

This statement has led several authors to consider alternative approaches based on the processing of heterogeneous strain fields that cannot be directly inverted (arising in mechanical tests which are sometimes called statically undetermined tests). In this case, both limitations can be overcome. Indeed, if the test is well designed, the applied loading gives rise to a heterogeneous strain field which involves the whole set of constitutive parameters. Their simultaneous identification is therefore possible provided that the response is measured on enough data points with some suitable non contact measurement technique and that a robust identification strategy extracting the parameters from this type of measured data is available. Full-field measurement techniques such as digital image correlation, speckle and moiré interferometry or grid method have recently spread in the experimental mechanics community, thus leading their users to wonder about the use of such measurements for identification purposes. The availability of identification strategies enabling the extraction of the constitutive parameters is in fact a key issue in such approaches since there is generally no closed-form solution allowing a direct link between measurements and unknown parameters. Different approaches have been proposed in the recent past among which model updating techniques and the so-called virtual fields method. This section will give the main features of these approaches and Chapter 3 will illustrate their relevancy through a review of some recent examples of application. One of the objectives is also to raise interest from the mechanics of materials community that is gradually becoming more and more aware of the potential of such approach to address difficult mechanical characterization issues.

2.2. Principal Methods of Resolution

2.2.1. Direct Inversion

Experimentalists have always tried to develop appropriate fixtures that provide the stress conditions under which they want to characterize their materials. If one assumes that this goal has been reached, the stress tensor is known at given points across the tested specimen. Measuring the strain tensor at the same points and using some further assumptions (plane stress, plane strain…), it is possible to plot the strain/stress curves of the material and to deduce the constitutive equations that link the stress and strain tensor straight forwardly. Consequently, there is no need to solve an inverse problem. Function g in Eq. (1) is bijective and it is known explicitly. It is simply inverted for determining the unknown constitutive parameters. This approach is called *direct inversion*.

Regarding isotropic materials, tensile tests are often sufficient for mechanical identification, and they are well suited to the use of *direct inversion*. This is more difficult for anisotropic materials, with coupling terms in the constitutive equations, especially in elasto-plasticity and elasto-visco-plasticity. Some references can easily be found about the tests that can still be used for characterizing the mechanical properties of materials using *direct inversion*. Thus, these tests will not be developed here. The most common ones are: the simple tension test, the compression test, the three-point bending test, the four-point bending test, the biaxial tensile test, the simple shear test, the Iosipescu shear test, off-axis tensile tests for composite materials…

Direct inversion requires quasi-static conditions to be satisfied. The existence of inertial effects in the previous tests complicates the relationships between the measurements and the constitutive parameters. Direct inversion may still be used in the experiment of Hopkinson bars with a few assumptions, but several authors have already pointed out the irrelevancy of these assumptions, requiring the resolution of the inverse problem with model updating techniques or other alternatives [Kajberg et al., 2004b].

2.2.2. Model Updating Techniques

In problems where the inverse of function g in Eq. (1) is not known explicitly, the most commonplace method used to identify parameters driving mechanical constitutive equations of materials is known as finite element model updating (FEMU). The principle of this technique consists in building up a finite element model of the mechanical test under study using information on geometry and boundary conditions. Thanks to this model, it is possible to compute explicitly a discretized model of the response corresponding to a given set of parameters, this modeled response being denoted $g(p)$ in Eq. (1). The observed response from the experiment, denoted as u, in Eq. (1), may be displacements, strains, forces, etc... Let p denote the set of material parameters to extract and p^0 is the initial estimate of p necessary to start the optimization scheme. A cost function is then built up consisting in some distance between u and $g(p)$. Starting from p^0, directions for minimizing the cost function are sought iteratively until the predefined convergence criterion is reached.

The weighted least-squares format is very commonly used, but not mandatory. For instance, prior information on u may be expressed using statistical approaches, e.g. based on Bayesian techniques or simply by introducing a penalty-like term in the objective function [Tarantola et al., 2005]. More sophisticated cost functions based on variational principles, such as the error in constitutive relation, have also been used occasionally with success [Bonnet and Constantinescu, 2005].

Depending on the choice of the measurable quantity being used for identification purposes, a very wide range of situations can be considered. In fact, it is desirable, but by no means necessary, that measured quantities be a field quantity, because in principle the FEMU technique may be applied to any kind of over-determined data. In practice model updating can be performed, without any further assumption, onto a set of data collected on the boundary of the specimen only. Finally, it must also be mentioned that the discretized model may be set up on the basis of other numerical techniques such as the finite difference method or the boundary element method or, occasionally, closed-form solutions.

A large number of examples can be drawn from the literature regarding the application of FEMU when experimental observations consist in full-field data. Several studies have dealt with the identification of bending rigidities in thin specimens. Some authors developed test design approaches based on finite element sensitivities [Le Magorou et al., 2002]. This enabled them to exhibit plate bending test configurations such that the plate response was influenced by all the stiffness parameters to be identified. Experimental implementation of such bending tests has been performed on structural wood-based panels. Interesting studies can be found as well for the mechanical behavior of biological tissues. For instance, some authors applied a shadow moiré technique to measure the hyperelastic properties of tissue-like membranes [Genovese et al., 2006]. FEMU has also been applied to the identification of parameters driving the elasto-plastic behavior of metals. After a first series of experiments on usual rectangular tensile specimens (aluminum, steel), tensile tests on notched or drilled specimens have been performed by the concerned authors in order to give rise to heterogeneous states of stress [Meuwissen et al., 1998]. Displacements are usually measured using digital image correlation (DIC) and various constitutive models have been tested (linear and non linear hardening). Generally, the boundary conditions of the finite element model are provided by the DIC patterns located at the boundary of the measurement zone (near the clamps).

Eventually, in view of the wide range of applications, it can be concluded that FEMU is a very flexible technique. Its main drawbacks are related to the convergence issues, and consequently to the time that is sometimes required for obtaining a solution (23 hours reported in a recent paper for only a few parameters [Kajberg et al, 2004]). A certain number of points have still to be improved for addressing these issues:

- the choice of the test configuration: this is a crucial issue where lots of progresses have to be made. Most of the time, the tests carried out by the researchers are either derived from usual testing configurations or invented using some common sense rules. Apart from marginal studies, no attempt at some systematic test design has been carried out to ensure maximum sensitivity of the technique to the unknown parameters.
- the choice of the experimental input data: the FEMU procedure will obviously be very sensitive to the choice of the experimental input data d. In particular, when considering heterogeneous strain fields, the question of the spatial resolution of the optical technique compared to the strain gradients is central. Moreover, the question also arises as to which data are to be used: displacements or strains and/or force, and over which part of the specimen.
- the choice of the objective function: the choice of the cost function seems to play an important role in the performance of the technique.
- the choice of the optimization algorithm, in terms of convergence time and robustness.
- the sensitivity to measurement errors: the goal would be to discriminate between identification errors coming from measurement uncertainty and errors in the model (boundary conditions, constitutive equations).

2.2.3. The Virtual Fields Method

This approach is applicable to situations where the field of observables u is experimentally known on the whole part of the solid to be characterized. No limitation is put on the loading conditions, which are assumed to be known only through the possible measurement of a resulting load. One seeks to identify parameters governing an assumed constitutive model for the material.

Assuming that the observables cover the whole domain of interest, denoted as Ω, and that they satisfy Eq. (2), the principle of virtual work (PVW) may take the form:

$$\int_{\Omega} M(u,p).u^{*}\,d\Omega = 0 \tag{4}$$

where u^* is an arbitrary virtual displacement field, which is constrained by the admissibility condition (continuity, differentiability, nullity over boundaries with prescribed displacement values).

The virtual fields method (VFM) basically consists in exploiting Eq. (4) with particular choices of virtual fields, tailored to the specific identification problem at hand [Grédiac et al., 2006]. First a constitutive model is chosen, yielding an expression of M in terms of p and the known observables u in the integral of Eq. (4). For instance, one has $M(u,p) = K(p).u - F$ like in Eq. (3) if elastic moduli are to be identified.

It must be emphasized that most full-field measurement methods provide in-plane displacement components only over the external surface of solids. 3D bulk measurement systems are under progress but their use for material identification remains marginal [Avril et al., 2008]. Specimens must therefore be chosen such that deformations in the solid can be analytically related to deformations on the surface. This is in particular the case for plane stress, plane strain, or bending of thin plates. As a consequence, the integral in Eq. (4) can be computed as a function of p for any chosen virtual field. A system of equations (generally non-linear, but sometimes linear if the behavior is linear elastic for instance) is derived and solved, providing the unknown parameters p with a remarkably low number of computational operations.

Parameter identification for the following classes of constitutive models have been considered in studies carried out during the recent past: linear anisotropic elasticity (in-plane and bending properties of anisotropic plates, either in terms of stiffnesses or invariant combinations of these stiffnesses), nonlinear anisotropic elasticity and, more recently, elasto-plasticity and elasto-visco-plasticity. Most of these studies have been carried out in quasi-static conditions. On a few occasions, dynamic properties have been identified [Giraudeau et al., 2006], for which inertial effects must be included in the model of Eq. (3) and in the integral of Eq. (4).

Then, one or several virtual fields are chosen. Each new virtual field introduced into Eq. (4) yields a new equation. The constitutive parameters are then sought as solutions to a set of such equations. Obviously, the number of equations thus obtained must be larger or equal to the number of parameters to be identified. The construction of the virtual fields is a key issue of the method. Three main different possibilities have been investigated so far:

i. the easiest one is to construct virtual fields by hand and to consider that the same expressions apply over the whole specimen, without any other particular rule than choosing kinematically admissible fields. For instance, in the case of anisotropic elasticity, as many independent virtual fields as unknown parameters are chosen, leading to a linear system where the constitutive parameters are unknown. When complex moduli are to be determined, as in the case of damping characterization, virtual fields can be complex as well [Giraudeau et al., 2006]. This finally leads to the direct determination of the unknown parameters by inverting the final system of linear equations. In this case, the virtual fields are often chosen as polynomials for the sake of simplicity [Grédiac et al., 2006];

ii. the basic approach presented previously was improved thanks to the automatic construction of the polynomial virtual fields. The procedure only applies in the cases where the PVW leads to a linear equation of the unknown constitutive parameters. To find one of these unknowns, the idea is to determine virtual fields such that the coefficient of this unknown in the equation is one and the remaining coefficients are zero. The unknown is therefore directly identified in this case with the virtual work of the external loading. Such virtual fields are referred to as *special*. They behave in fact like filters which directly extract the unknown parameters from the measured heterogeneous strain fields. It can be shown that special virtual fields are not unique but infinite. This extra freedom can be used to find some more relevant special virtual fields. Relevancy is assessed here in term of sensitivity to noisy data. An efficient procedure has been proposed recently. A basis of independent functions is first given to define the virtual fields. Then the procedure directly provides the unique virtual field which leads to the lowest sensitivity to noise of the identified parameters;

iii. the virtual fields can also be defined piecewise instead of being expressed by the same relation (polynomials usually) over the whole specimen. Defining sub-regions in the specimen and constructing virtual fields in each of these sub-regions induces more flexibility in their definition. In this case, lower-degree polynomials are used as shape functions in each sub-region. Indeed, when virtual fields are defined with the same expression over the whole specimen, higher-degree polynomials may be required and it was observed that such higher-degree polynomials magnify the adverse effect of noise on identified parameters. The only condition here is that the piecewise virtual fields satisfy the C^0 continuity between each sub-region. The C^1 continuity is not required here: a C^1 discontinuity does not induce any problem in the calculations, contrary to the similar discontinuity which occurs in the finite element method when actual displacement fields are approximated with piecewise functions.

Three other features concerning the construction of the virtual fields must be underlined.

- On the one hand, the loading distribution, which is involved in the virtual work identity in Eq. (4) through forces F, is often difficult to measure in practice. On the other hand, the resulting force is usually measured. In such cases, the virtual fields can be defined so that the virtual work involves only the resulting force. This property is obtained by prescribing a constant virtual displacement over the boundary where tractions are applied, along the direction of this resulting force.

- The virtual work of the tractions at the boundary can be cancelled out by selecting virtual fields vanishing over the boundary. This strategy seems to be unavoidable when very local properties are to be identified. Eq. (4) then involves only one non-zero integral under static conditions. One of the constitutive parameters at least must be considered as known *a priori* to avoid the final system of linear equations to be homogeneous. If not, then only stiffness ratios can be determined.

- Third, 3D integrals must be computed in practice, thus leading the deformation field to be known within the solid. From a theoretical point of view, this feature is undoubtedly a limitation of the VFM when 2D measurements only are available. In practice however, relevant assumptions can be used to deduce the displacement/strain fields within the solid from measurements collected on the external surface. For instance, mechanical tests are often carried out on beam- or plate-like specimens. In theses cases, assumptions of constant or linear strain distribution through the thickness of the specimens are sound, thus leading the VFM to be suitable for solving a wide range of problems. All the strategies described above have been applied in various cases of parameter identification, either with numerical data for validation purpose or experimental data for practical applications.

Compared to FEMU methods, the VFM appears to be less flexible because the requirement of full-field data is a limitation. Actually, full-field data never exist in practice. Optical or tomographic methods can provide dense data across a given surface or volume, but field u must always be reconstructed afterwards for applying the VFM, as this field is required in Eq. (4). This step of field reconstruction is often critical for the VFM, because, even if field u can always be reconstructed in practice from non-dense noisy data, robustness issues may arise during the identification process if the spatial resolution of the measurements is not sufficient.

The robustness of the VFM has been thoroughly investigated for problems of elastic moduli identification. It has been shown that the uncertainty of the parameters identified with the VFM when a white noise is added to the data depends on the choice of a relevant set of virtual fields. Optimal virtual fields exist, thus minimizing the uncertainty and providing the ‘‘maximum likelihood solution’’.

It has also been proved for problems of elastic identification and when field u is available, that FEMU approaches actually yield equations similar to the ones derived from the VFM, but with non-optimal sets of virtual fields [Avril and Pierron, 2007]. Therefore, the FEMU approaches do not provide the ‘‘maximum likelihood solution’’. However, the uncertainty of FEMU approaches varies dramatically with the cost function to minimize. On one hand, the FEMU approach based on a least-squares ‘‘displacement gap’’ minimization yields equations which are very close to the ones of the VFM approach and therefore, its uncertainty is almost the same as the VFM one. On the other hand, it is shown that other approaches based for example on the ‘‘constitutive equation gap’’ minimization or the ‘‘equilibrium gap’’ minimization may provide biased solutions.

Moreover, very fast algorithms, converging in only two iterations, have also been devised thanks to the formalism of the VFM [Avril and Pierron, 2007]. This aspect is crucial because it was concluded in Section 2.2.2 that the main drawback of FEMU methods is convergence time. This drawback is enhanced with the increase of the number of data to process. Eventually, this leads to the following conclusion:

- when only a few data are available (not full-field or full-field with a low density), a FEMU method is better suited because it may be robust even without the whole field of observables;
- when full-field data with high density are available, the VFM is better suited as it converges quickly and it is very robust.

The study previously mentioned [Avril and Pierron, 2007] about comparing the performances of FEMU and VFM only concerns the identification of elastic moduli. As far as elasto-plastic [Avril et al., 2008a] or visco-elasto-plastic [Avril et al., 2008c] properties are concerned, convergence time seems to be even more crucial. Forward problems have to be solved a great number of times in FEMU method for solving such non-linear inverse problems, which is quite penalizing. On the contrary, the VFM has proved to be both rapid and robust. Therefore, the conclusion is that the VFM is the approach most suited to the identification of elasto-plastic or elasto-visco-plastic properties from full-field measurements.

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Biographical Sketches

Fabrice Pierron was born in Strasbourg, France, on April 6th 1966. He obtained an engineering degree from École Nationale Supérieure d'Electricité et Mécanique (ENSEM, Nancy, France) in 1989, and received a PhD from Lyon University (France) in 1994.

After an Assistant Professor position at the École Nationale Supérieure des Mines de Saint-Étienne (France) from 1994 to 1999, he moved to École Nationale Supérieure d'Arts et Métiers (ENSAM, Châlons-en-Champagne, France), now Arts et Métiers ParisTech, on a Professor position where he created and leads the Mechanical Engineering and Manufacturing Research Group (LMPF, 20 people, www.lmpf.net). He is an expert in mechanical testing of materials and in composites but has been focusing his research on the use of novel identification strategies based on full-field measurements and heterogeneous tests for about ten years. In particular, he has been instrumental in the development of the Virtual Fields Method (www.camfit.fr). He has published over 50 journal papers (with a current h-factor of 13) and coauthored more than 150 communications in National and International conferences.

Prof. Pierron is a member of the French Mechanics Society (AFM), the French Composites Society (AMAC), the French Mechanics of Materials Society (MECAMAT), the Society for Experimental Mechanics (SEM), the British Society for Strain Measurements (BSSM) and the European Society for Experimental Mechanics (EURASEM).

Stéphane Avril was born in Saint-Etienne, France, on August 9th 1976. He obtained an engineering degree from "École Nationale Supérieure des Mines" (ENSM, Saint-Etienne, France) in 1999, and received a PhD from Saint-Etienne University (France) in 2002. After obtaining a first position in 2003 at "Arts et Métiers ParisTech" (ENSAM, Châlons-en-Champagne, France), he moved to ENSM Saint-Etienne in 2007, on a position of Assistant Professor in biomechanics. He is an expert in inverse methods, in particular regarding the experimental characterization of solids with full-field measurements. Most of his current research is focused on the in-vivo mechanical behavior of human soft tissues. He has published 19 journal papers and co-authored more than 50 communications in National and International conferences.