CELESTIAL MECHANICS: FROM ANTIQUITY TO MODERN TIMES

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Summary

Celestial Mechanics has followed the development of humankind from antiquity to space exploration age. After a short review of the cosmological models dating back to ancient Greeks, we proceed to present the main achievements which led to modern Celestial Mechanics. First, we describe Kepler's laws which explain how planets move around the Sun, thanks to Newton's gravitational force. Despite the fact that they were discovered in the XVII century, these laws provide interesting tools for managing a spacecraft trajectory, like the so–called Hohmann transfers and the gravity assist technique. Then we present perturbation theory developed in the XVIII century, which is an extremely important tool in Celestial Mechanics: for example, it led to the discovery of Neptune, to the computation of the perihelion's precession as well as to accurate lunar ephemerides. Stability results can be obtained thanks to the outstanding theories developed in the XX century by Kolmogorov, Arnold, Moser (KAM theory) and Nekhoroshev, that we shortly present, together with some of their applications to Celestial Mechanics.

1. Introduction

Celestial Mechanics is devoted to the study of the motion of the celestial bodies which influence each other, mainly due to the gravitational law. This discipline has born and developed together with humankind: the computation of the succession of seasons was of fundamental importance for the survival of human race, the prediction of the phenomena of the sky like eclipses is as old as Babylonian people, the determination of the position on a ship in the sea relied on the knowledge of the position of stars and planets. These are just a few reasons which led scientists to study the dynamics of the celestial bodies and to develop cosmological models. During antiquity these models were based on the assumption that the Earth was at the center of the cosmos and that all bodies (Sun and planets included) should move on spheres and circles, being these the most symmetric and perfect geometrical shapes. This cosmological model survived for several centuries, until Copernicus made his revolution by dethroning the Earth at the center of the cosmos, replacing it by the Sun and letting the Earth move around the Sun. The Copernican revolution was followed by giants of science like Galileo, Kepler and Newton.

During the last centuries, Celestial Mechanics has profited of the observational discoveries to test the theoretical results and, conversely, astronomy has used the mathematical theories to make new observations and discoveries. The interplay between observations, technological advancements and Celestial Mechanics is even more evident when looking at the last 100 years, as shown in the timeline below.

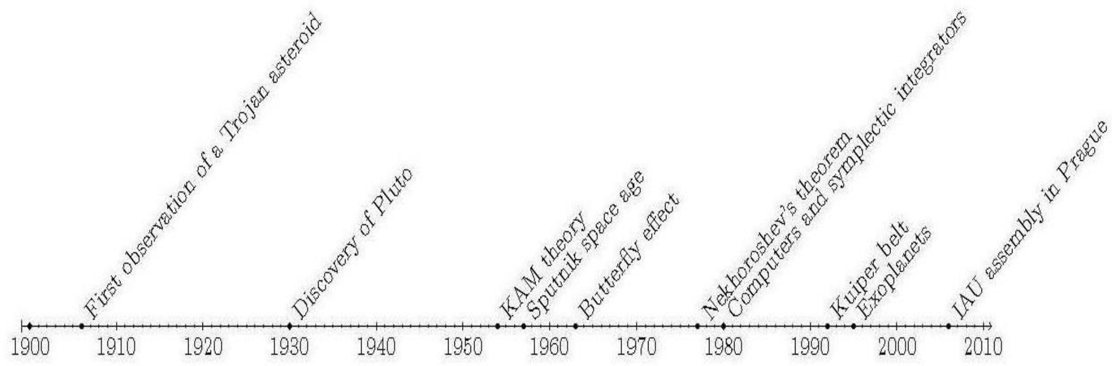

Figure 1. Timeline showing interplay between observations, technological advancements and Celestial Mechanics

When Lagrange discovered the triangular equilibrium positions, he thought that it was a nice mathematical result without any physical application. On the contrary, the first asteroid in the triangular position, belonging to the so–called *Trojan* asteroids, was discovered in 1906 and nowadays several space missions take advantage of the collinear and triangular Lagrangian points. Perturbation theories, started in the XVIII century, led to the discovery of Neptune in 1846 and later contributed to the discovery of Pluto in 1930. Around the middle of the XX century, outstanding mathematical results, motivated by the investigation of the stability of the solar system, were developed by A.N. Kolmogorov, V.I. Arnold, J. Moser (the so–called KAM theory) and later by N.N. Nekhoroshev. Chaos theory, originally discovered by H. Poincaré during his studies on the three–body problem, started its golden age when computers appeared. The *butterfly effect*, originated by computer simulations on differential equations describing a meteorological system, becomes the paradigm of chaos: a small change of the initial conditions (like a flap of the butterfly's wings) might provoke a big challenge (like a tornado at several miles of distance).

The era of the space missions started with the launch of the Sputnik in 1957; this event opens a new branch of Celestial Mechanics, called Astrodynamics. The subsequent advent of faster and cheaper computers, as well as the development of symplectic integrators, gives another big impulse to Celestial Mechanics. Finally, the last decade of the XX century was marked by two astronomical epochal discoveries: the Kuiper belt in 1992 and the first extra–solar planetary system in 1995. These discoveries have provoked a new view of the solar system and of its dynamical behavior, culminated with the IAU (International Astronomical Union) assembly in 2006, which excluded Pluto from the list of planets and re-defined the whole solar system.

2. From Ptolemy to Copernicus

Thanks to their geometrical symmetry, circles and spheres dominated the cosmological models developed during antiquity. Ptolemy's cosmology was in fact based on the assumption that the Earth was located at the center of the cosmos and that the dynamics of the celestial bodies could be explained by using suitable combinations of circles, known as epicycles and deferents. Fourteen centuries were needed to leave the Ptolemaic viewpoint and to embrace the Copernican model.

2.1. Epicycles and Deferents

The theories of ancient Greeks were dominated by the idea of providing a scientific proof of the perfection of Nature. Due to their symmetry, the most important geometric forms are the circle (in the plane) and the sphere (in the space); as a consequence, ancient theories were typically based on these forms by assuming a spherical structure of the universe as well as a circular motion of the celestial bodies. The use of spheres to represent the cosmos provides the best tool to get the geometric perfection of the universe. The sky appears like an enormous sphere with center in the Earth; on this sphere one can find the *fixed stars*, namely the bodies whose positions appear unaltered in time. The big sphere representing the sky is formed by crystalline and unbreakable material, in contradiction to the materials which form the known worlds according to the Aristotelian physics: air, water, earth and fire. The Sun, Moon and planets live within this sphere and they move on concentric and transparent shells. Within each planetary shell, the motion of the celestial bodies is represented by circles, each one run with constant velocity. According to these philosophical–scientific lines of thought, Ptolemy (ca. 100–170 A.C.) provided his ideas in the impressing opera titled "*Almagest*", which means "The Great Treatise". Following Aristotle (384–322 B.C.) and the other predecessors, he builds up a model of the universe which appears to be geometrically perfect. Ptolemy's cosmology is based on the following assumptions:

- The universe has a spherical shape;
- The Earth is a sphere;
- The Earth is at the center of the universe and it does not move;
- All other celestial bodies move on spherical shells centered on the Earth;
- The size of the Earth is negligible compared to the distance to the fixed stars.

The Earth is placed at the center of the cosmos and it is surrounded by a first spherical shell on which the Moon moves; going farther from the Earth, on the next shell one finds Mercury and at higher distances the shells of Venus, the Sun, Mars, Jupiter and Saturn are placed. Finally, one finds the shell corresponding to the fixed stars. The

Ptolemaic vision persists during fourteen centuries, despite the fact that the astronomical observations showed several discrepancies with the cosmological theory. In fact, the motion of some planets, as seen from the Earth, exhibits irregularities: an observer located on the Earth's surface has the perception of the planet as moving in one direction on the celestial sphere, then stopping and coming back on a path along a retrograde direction. Such behavior will be later explained as the result of a combined effect of the motion of the Earth and of the planet around the Sun. Nevertheless, in antiquity such explanation was not consistent with the assumption of a steady Earth at the center of the cosmos.

Despite the fact that the astronomical observations contradict the assumption of a uniform, circular motion of the planets around the Earth, the idea of a harmonic dynamics of Nature, based on the perfect regularity of a circular orbit run with uniform velocity, dominated the theories of planetary motions within Greek culture. In order to keep the assumptions unaltered, but at the same time to have a consistency between theory and observations, Greek scientists developed astronomical models based on suitable combinations of circular trajectories on which a uniform motion takes place. First, Apollonius from Perga (ca. 262 BC –- ca. 190 BC), a Greek geometer and astronomer, invented a model according to which the planets were moving on circular orbits whose center did not coincide exactly with the center of the Earth. Since such theory was not sufficient to explain the anomalies shown by the astronomical observations, in particular the variation of the velocities, stations and retrogradations, Apollonius modified his model by assuming that the *anomalous* planet was rotating with uniform motion on a circle, called *epicycle,* whose center was moving on another circle, named *deferent,* with center in the Earth (compare with Figure 2a).

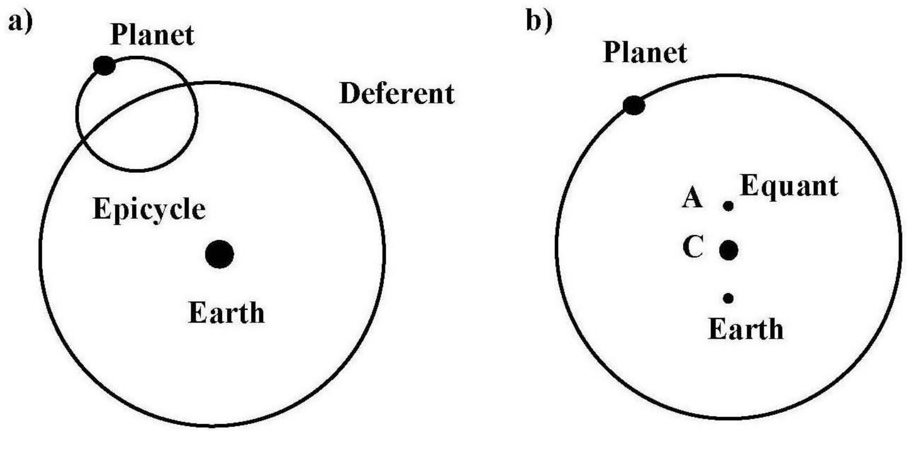

Figure 2. a) The model of epicycles and deferents developed by Apollonius from Perga. b) The theory of equants developed by Ptolemy.

Inheriting the conjectures by Apollonius and having at disposal the important astronomical observations performed by Hypparcus (ca. 190–120 BC), in his "*Almagest*" Ptolemy introduced a variation of the model based on epicycles and deferents. More precisely, he assumed that a planet was moving on a circle with center C on a trajectory run with variable velocity; he defined a point A, called equant, whose distance from the center is equal to that of the Earth from C (see Figure 2b). Ptolemy's model assumed that the motion of the planet occurred in such a way that on observer located in A could see the planet to revolve with constant velocity with respect to the equant A. As a consequence, an observer located on the Earth sees the planet moving slowly when it is in the region closer to the equant and moving faster otherwise. It is relevant to underline that in this model the Earth is still kept fixed. According to Ptolemy's model of the planetary motions, the variation of velocities of the planets as observed by the Earth is justified by the construction with epicycles, deferents and equants. The common ground of each model is the use of a suitable combination of circles, in order to guarantee the philosophical belief of a perfect symmetry and harmony of the cosmos.

2.2. The Copernican Revolution

The breakthrough of the Ptolemaic model came through the opera "*De Revolutionibus Orbium Caelestium*" by Nicolaus Copernicus (1573–1543). Indeed, the idea that the Earth could not be at the center of the cosmos was already predicted in antiquity by Aristarchus of Samos (310 BC –- ca. 230 BC), who placed the Sun at the center of the homocentric spheres. Nevertheless, the spectacular contribution of Copernicus consisted in developing a mathematical theory allowing us to explain the dynamics of the planets in compliance with the astronomical observations. The main assumption was that the planets were rotating around the Sun, instead than around the Earth as conjectured by the Ptolemaic theory. Copernicus' model allows us to split the solar system in internal planets (Mercury and Venus, observable at sunshine and early morning) and external planets (from Mars on).

If we admit that also the Earth can move in the sky, the explanation of the anomalies concerning the orbits of the planets, like the variation of velocities, stations and retrogradations, can be interpreted at the light of a heliocentric model. Using Copernicus words: "*We must conclude, then, that their uniform motions [of the planets] appear to us as irregular either because they take place around different axes, or else because the Earth is not at the center of their circles of revolution*".

The Sun takes the major role and it is placed at the center of the universe. In "*De Revolutionibus Orbium Caelestium*" Copernicus states: "*In the center rests the Sun. For who would place this lamp of a very beautiful temple in another or better place than this wherefrom it can illuminate everything at the same time*".

Copernicus finally gives the explanation for the dynamics of the solar system: by using his heliocentric model, one can conclude that the planets move around the Sun and that the Earth itself orbits around the Sun, taking one year to make a full revolution.

2.3. The Astronomical Revolution

Due to the religious and political censorship, the ideas of Copernicus were considered a mere theoretical hypothesis. Nevertheless the "*De Revolutionibus Orbium Caelestium*" was the beginning of a new era, where some far–sighted scientists like Giordano Bruno, Galileo Galilei and Johannes Kepler understood the validity of the Copernican model in order to let science progress in that direction.

A fundamental role was played by the Danish astronomer Tycho Brahe (1546–1601). Although he was not a supporter of the Copernican system, Brahe contributed to the scientific progress by performing an intense observational campaign of planets and stars. Thanks to his brilliant discovery of a supernova in 1572, the king of Denmark and Norway supported economically a project to build an astronomical observatory in the Hveen island. The new observatory was called *Uraniborg;* there Brahe classified several celestial bodies with an astonishing accuracy. Beside stars, Brahe observed every day the position of the Sun and the planets. His data formed the basis for the successive researches of his young collaborator, the German mathematician and astronomer Johannes Kepler (1571–1630).

The happily years of astronomy and physics continued with Galileo Galilei (1564–1642). He gave a great impulse for the development of a new way to approach the scientific research. Galileo studied the physics of Nature in all its aspects and encoded it using mathematical laws. For the first time in humankind, Galileo used a telescope (just discovered at his times) to observe the sky.

He started an incomparable observational campaign, which leaded to astonishing discoveries. Among the others, Galileo studied the Moon which appeared to be characterized by mountains and craters, he classified several stars, he observed the Milky Way and Sun spots, he discovered the main satellites of Jupiter (Io, Europa, Ganymede and Callisto, later named *Galileian satellites*), he determined the phases of Venus and he noticed around Saturn some bulges close to the planet (with better instruments they were later distinguished as being the famous Saturn's rings).

In his masterpiece titled the "*Dialogue Concerning the Two Chief World Systems*", Galileo supports the Copernican model by providing scientific arguments based on his astronomical observations. His results and the new approach to the study of Nature allowed Kepler and Newton to provide a comprehensive understanding of the laws governing the planetary motions.

3. Kepler's Laws and Hohmann Transfers

3.1. Kepler's Laws

The changeover of the scientific progress continues with Johannes Kepler; he endorses the Copernican theory and uses the data provided by the monumental work of Tycho Brahe in order to formulate, without even using a telescope, some laws describing the motion of the planets around the Sun. The first problem encountered by Kepler is the necessity to neglect the assumption that the planetary orbits are circular. Being reluctant to make this hypothesis, in his work titled "*Mysterium Cosmographicum*" Kepler develops a complicated formulation of the planetary motions, still constrained to a circular orbit, but assuming the heliocentric model. However, the data provided by Brahe showed that the distance of Mars from the Sun was in some points less than that obtained assuming that Mars moves on a circular orbit. Kepler understood that the initial assumptions were wrong and after 70 attempts made during 5 years, he was able to formulate three fundamental laws which clarify how planets move around the Sun. The first two laws were stated in the work "*Astronomia Nova*", while the third law was given in the "*Harmonices Mundi*".

Over several years Kepler studied Mars, whose orbit is elliptic, but almost circular, being the eccentricity very small. A very high degree of precision was then necessary in order to characterize the trajectory of Mars. Although he was using poorly accurate instruments, Kepler was able to determine that the orbit of Mars is not circular. The characterization of the planetary orbits is the content of *Kepler's first law*; however such law does not provide information about the velocity with which the orbit is run and it does not yield the dimension of the ellipse with respect to the time needed by the planet to run a full trajectory around the Sun. These quantities are provided by the second and third laws, thus completing the mosaic of the planetary dynamics. In fact, Kepler's second law states that the planet is faster when it is closer to perihelium (namely, the point on the orbit closer to the Sun) and it is lower at aphelion (the point of the orbit farther from the Sun). Translated in geometric terms, Kepler's second law is equivalent to state that the planet spans equal areas during the same time intervals.

Once he discovered the shape of the orbit and the size of the velocity along the trajectory, Kepler determined the relation between the time necessary to run an orbit and its size: he established a proportionality relation between the square of the period of revolution and the cube of the semimajor axis. The consequence of this law is very important: the more is the distance of the planet from the Sun, the largest is the time needed to run a full orbit.

To summarize, the formulation of the three Kepler's laws is the following:

I Law: *The orbit of a planet around the Sun is elliptical and the Sun is placed at one of the two foci.*
II Law: *The planet describes equal areas in equal time intervals.*
III Law: *The square of the period of revolution is proportional to the cube of the semimajor axis.*

As stated before, Kepler's laws show that the solution of the two–body problem is an ellipse, provided that the total mechanical energy is negative. One can prove that the two–body problem admits also parabolic orbits, whenever the energy is zero, and hyperbolic trajectories for positive energies.

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Biographical Sketch

Alessandra Celletti (born in 1962 in Rome, Italy) received her Master degree in 1984 at the University of Roma La Sapienza under the supervision of G. Gallavotti and her PhD at the ETH in Zürich, Switzerland, in 1989 under the supervision of J. Moser and J. Waldvogel. She is currently professor of Mathematical Physics at the University of Rome Tor Vergata. She was president and founder of the Italian Society of Celestial Mechanics and Astrodynamics, and director of the master course on "Space Science and Technology" at the University of Rome Tor Vergata. In 2017 she has been elected editor-in-chief of "Celestial Mechanics and Dynamical Astronomy". She is member of the "Celestial Mechanics Institute", vice-president of the Organizing Committee of the "International Astronomical Union Commission 7: Celestial Mechanics and Dynamical Astronomy", member of the EMS/EWM Scientific Committee. Her research interests concern Celestial Mechanics and Dynamical Systems, with particular reference to KAM theory for conservative systems, KAM theory for dissipative systems, computer–assisted proofs, rotational dynamics, the three–body problem and the stability of the Lagrangian solutions.