THE N-BODY PROBLEM

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Summary

We introduce the N-body problem of mathematical celestial mechanics, and discuss its astronomical relevance, its simplest solutions deduced from the two-body problem (called homographic motions and, among them, homothetic motions and relative equilibria), Poincaré's classification of periodic solutions, symmetric solutions and in particular choreographies such as the figure-eight solution, some properties of the global evolution and final motions, Chazy's classification in the three-body problem, some non-integrability results, perturbations series of the planetary problem and a short account on the question of its stability.

1. Introduction

The problem is to determine the possible motions of N point particles of masses $m_1,\ldots,m_N$, which attract each other according to Newton's law of inverse squares. The conciseness of this statement belies the complexity of the task. For although the one and two body problems were completely solved by the time of Newton by means of elementary functions, no similar solution to the N-body problem exists for $N \geq 3$.

The N-body problem is intimately linked to questions such as the nature of universal attraction and the stability of the Solar System. In the introduction of the *New Methods of Celestial Mechanics*, Poincaré suggested that it aims at solving "this major question to know whether Newton's law alone explains all astronomical phenomena". But since the N-body problem ignores such crucial phenomena as tidal forces and the effects of general relativity, this model is now known to be quite a crude approximation for our Solar System. So it is not useless in this introduction to give some brief account of how the N-body problem has become a central piece of celestial mechanics and remains so. For further background, we refer to *Celestial Mechanics: From Antiquity to Modern times*.

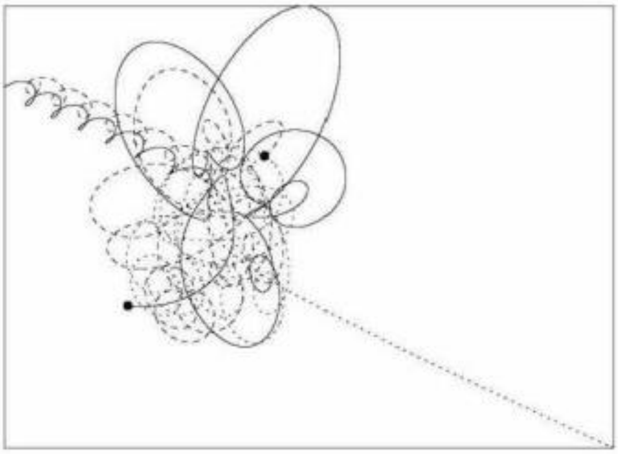

Figure 1. A solution of the plane three-body problem, starting from a double collision and leading to a hyperbolic escape

Hook's and Newton's discovery of universal attraction in the XVII century dramatically modified our understanding of the motion of celestial bodies. This law masterly reconciles two seemingly contradictory physical principles: the principle of inertia, put forward by Galileo and Descartes in terrestrial mechanics, and the laws of Kepler, governing the elliptical motion of planets around the Sun. In an additional *tour de force*, in his *Philosophiae naturalis principia mathematica,* Newton also estimated the first order effect on Mars of the attraction of other planets.

The unforeseen consequence of Hook's and Newton's discovery was to question the belief that the solar system be stable: it was no longer obvious that planets kept moving immutably, without collisions or ejections. And symmetrically, the question remained for a long time, whether universal attraction could explain the irregularities of motion, due to the mutual attraction of the various celestial bodies, observed in the past. A two-century long competition started between astronomers, who made more and more precise observations, and *geometers*, who had the status and destiny of Newton's law in their hands. Two main mysteries kept the mathematical suspense at its highest: the motion of the Moon's perigee, and the shift of Jupiter's and Saturn's longitudes, revealed by the comparison between the observations of that time and those which Ptolemy had recorded almost two thousand years earlier. The first computations of Newton, Euler and others were giving wrong results. But infinitesimal calculus was in its infancy and geometers, at first, lacked the necessary mathematical apparatus to understand the long-term influence of mutual attractions.

Regarding the Moon's perigee, Clairaut and d'Alembert understood that the most glaring discrepancy with observations could be explained by higher order terms. Thus the works of Euler, Clairaut, d'Alembert and others in the XVIII century constituted the Newtonian N-body problem as the description of solutions of a system of ordinary differential equations (see Section 2). The problem was given a major impulse when Lagrange transformed mechanics and dynamics into a branch of mathematical analysis, laying the foundations of differential and symplectic geometry.

In his study of Jupiter's and Saturn's motions, Laplace found approximate evolution equations, describing the average variations of the elliptical elements of the planets. These variations are called *secular* because they can be detected only over a long time interval, typically of the order of a century (= *secular* in Latin). Laplace computed the secular dynamics at the first order with respect to the masses, eccentricities and inclinations of the planets. His analysis of the spectrum of the linearized vector field, at a time when this chapter of linear algebra did not exist, led him and Lagrange to a resounding theorem on the stability of the solar system, which entails that the observed variations in the motion of Jupiter and Saturn come from resonant terms of large amplitude and long period, but with zero average.

We are back to a regular –namely, quasi-periodic (or *conditionally periodic*, according to the terminology of some authors)– model, however far it is conceptually from Ptolemy's ancient epicycle theory. Yet it is a mistake, which Laplace made, to infer the topological stability of the planetary system, since the theorem deals only with a truncated problem (see Section 10).

Around that time Euler and Lagrange found two explicit, simple solutions of the three-body problem, called *relative equilibria* because the bodies rigidly rotate around the center of attraction at constant speed. These solutions, where each body moves as if it were attracted by a unique fictitious body, belong to a larger class of motions, called *homographic*, parameterized by the common eccentricity of bodies; see Section 4, and *The Lagrangian Solutions*. Some mathematical and more global questions started to compete with the purely initial astronomical motivations. Recently, many new periodic orbits have been found, which share some of the discrete symmetries of Euler's and Lagrange's orbits in the equal-mass problem; see Section 6.

The theory of the Moon did not reach a satisfactory stage before the work of Adams and Delaunay in the XIX century. Delaunay carried out the Herculean computation of the secular dynamics up to the eighth order of averaging, with respect to the semi major axis ratio; as already mentioned, the secular dynamics is the slow dynamics of the elliptic elements of the Keplerian ellipses of planets and satellites.

The first order secular Hamiltonian is merely the gravitational potential obtained by spreading the masses of planets and satellites along their Keplerian trajectories, consistently with the third Kepler law. Delaunay mentioned *un résultat singulier*, already visible in Clairaut's computation: according to the first order secular system, the perigee and the node describe uniform rotations, in opposite directions, with the *same* frequency. This was to play a role later in the proof of Arnold's theorem (see *The Planetary N-Body Problem*), although higher order terms of large amplitude destroy the resonance.

At the same time as Delaunay, Le Verrier pursued Laplace's computations, but questioned the astronomical relevance of his stability theorem. In the XIX century, after the failure of formal methods due to the irreducible presence of small denominators in perturbation series generally leading to their divergence, Poincaré has drawn the attention of mathematicians to qualitative questions, concerning the structure of the phase portrait rather than the analytic expression of particular solutions, of the N -body problem.

In particular, Bruns and Poincaré in his epoch-making treatise, *The New Methods of Celestial Mechanics*, gave arguments against the existence of first integrals other than the energy and the angular momentum in the 3-body problem (see Section 9).

Some facts like the anomalous perihelion advance of the planet Mercury could only be explained in 1915 by Einstein's theory of general relativity. Classical dynamics thus proved to be a limit case of, already inextricably complicated but simpler than, Einstein's infinite dimensional field equations.

On the positive side, Poincaré gave a new impulse to the perturbative study of periodic orbits. Adding to the work of Hill and cleverly exploiting the symmetries of the three-body problem, he found several new families, demanding a classification in terms of *genre*, *espèce* and *sorte* (genre, species and kind); see Section 5.

In the XX century, followers like Birkhoff, Moser, Meyer have developed a variety of techniques to establish the existence, and study the stability, of periodic solutions in the N -body problem, and more generally in Hamiltonian systems: analytic continuation (in the presence of symmetries, first integrals and other degeneracies), averaging, normal forms, special fixed point theorems, symplectic topology. Broucke, Bruno, Hénon, Simó and others have quite systematically explored families of periodic orbits, in particular in the Hill (or lunar) problem.

Regarding perturbation series, a stupendous breakthrough came from Siegel and Kolmogorov, who proved that, respectively for the linearization problem of a one-dimensional complex map and for the perturbation of an invariant torus of fixed frequency in a Hamiltonian system, perturbation series do converge, albeit non uniformly, under some arithmetic assumption saying that the frequencies of the motion are far from resonances. Siegel's proof overcomes the effect of small denominators by cleverly controlling how they accumulate, whereas Kolmogorov uses a fast convergence algorithm, laying the foundations for the so-called Kolmogorov-Arnold-Moser theory; see Section 10, or *Classical Hamiltonian Perturbation Theory,* and *The Planetary N-Body Problem.*

Two discoveries have led to another shift of paradigm. First, came the discovery of exoplanets in the early 1990's. This confirmation of an old philosophical speculation has sustained the interest in extraterrestrial life. Many of these exoplanets have larger eccentricities, inclinations or masses (not to mention brown dwarfs), or smaller semi major axes, than planets of our solar system–and there seems to be billions of them in our galaxy alone. Are such orbital elements consistent with a stable dynamics? This wide spectrum of dynamical forms of behavior has considerably broadened the realm of relevant many-body problems in astronomy, and renewed interest in the global understanding of the many-body problems, far from the so-called planetary regime (with small eccentricities, inclinations and masses), and possibly with important tidal or more general dissipating effects.

The second discovery is mathematical. Nearly all attempts to find periodic solutions of the N-body problem by minimizing the action functional had failed until recently because collisions might occur in minimizers, as Poincaré had pointed out. Indeed, the Newtonian potential is weak enough for the Lagrangian action to be finite about collisions. In 1999 Chenciner-Montgomery overcame this difficulty and managed to prove the existence of a plane periodic solution to the equal-mass three-body problem, earlier found by Moore numerically, with the *choreographic* symmetry –a term coined by Simó, meaning that the bodies chase each other along the same closed curve in the plane. After this breakthrough, many symmetric periodic solutions have been found, theoretically and numerically. See Section 6.

For most of the topics in this Chapter, it is only possible to outline major results.

2. Newton's Equations and their Symmetries

The motion of N bodies is assumed to be governed by Newton's equations

$$\ddot{\mathbf{x}}_j = \sum_{k \neq j} m_k \frac{\mathbf{x}_k - \mathbf{x}_j}{\left\|\mathbf{x}_k - \mathbf{x}_j\right\|^3},\ j = 1,\ldots,N \qquad (1)$$

where $\mathbf{x}_j \in E = \mathbb{R}^d$ is the position of the j-th body in the d-dimensional Euclidean space, $\ddot{\mathbf{x}}_j$ its second time-derivative, m_j its mass, and $\|\cdot\|$ the Euclidean norm; the Euclidean scalar product of $\mathbf{x}_j$ and $\mathbf{x}_k$ will be denoted be $\mathbf{x}_j \cdot \mathbf{x}_k$. We have conveniently chosen the time unit so that the universal constant of gravitation, which is in factor of the right hand side, equals 1. The space dimension d is usually assumed less than or equal to three, but larger values may occasionally prove worth of interest.

Following Lagrange, the equations can be written more concisely

$$\ddot{\mathbf{x}} = \nabla U(\mathbf{x}),$$

where $\mathbf{x} = (\mathbf{x}_1,\ldots,\mathbf{x}_N) \in E^N$ is the configuration of the N points, U is the *force function* (opposite of the gravitational potential energy)

$$U(\mathbf{x}) = \sum_{j<k} \frac{m_j m_k}{\|\mathbf{x}_j - \mathbf{x}_k\|}, \tag{2}$$

and ∇U is the gradient of U with respect to the *mass scalar product* on E^N (in the sense that $\langle dU(\mathbf{x}), \delta\mathbf{x}\rangle = \nabla U(\mathbf{x}) \cdot \delta\mathbf{x}$ for all $\delta\mathbf{x} \in E^N$), the mass scalar product itself being defined by

$$\mathbf{x} \cdot \mathbf{x}' = \sum_{1 \le j \le N} m_j \left(\mathbf{x}_j \cdot \mathbf{x}'_j\right)$$

Introducing the *linear momentum* $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_N) \in E^N$, with components $\mathbf{y}_j = m_j \dot{\mathbf{x}}_j$, these equations can be put into Hamiltonian form ($|\mathbf{y}| = \sqrt{\mathbf{y} \cdot \mathbf{y}}$ for $\mathbf{y} \in E^N$) by saying that

$\dot{\mathbf{x}} = \partial_{\mathbf{y}} H$ and $\dot{\mathbf{y}} = -\partial_{\mathbf{x}} H$, where $H(\mathbf{x}, \mathbf{y}) = \frac{|\mathbf{y}|^2}{2} - U(\mathbf{x})$.

As a particular case of the general equations of dynamics, the equations of the N -body problem are invariant by the Galilean group, generated by the following transformations:

- Shift of time: $t' = t + \delta t, \quad \delta t \in \mathbb{R}$
- Shift of positions : $\mathbf{x}'_j = \mathbf{x}_j + \delta\mathbf{x}_0, \quad \delta\mathbf{x}_0 \in E;\ j = 1, 2, \ldots d$
- Space isometry: $\mathbf{x}'_j = \mathbf{R}\mathbf{x}_j, \quad \mathbf{R} \in O(E);\ j = 1, 2, \ldots N$
- Shift of velocities (or boost): $\dot{\mathbf{x}}'_j = \dot{\mathbf{x}}_j + \delta\dot{\mathbf{x}}_0, \quad \delta\mathbf{x}_0 \in E;\ j = 1, 2, \ldots N$.

The first three symmetries preserve the Hamiltonian and, according to Noether's theorem, entail the existence of first integrals, respectively:

- The energy $H \in \mathbb{R}$
- The linear momentum $\mathbf{P} = \sum_j \mathbf{y}_j \in E$
- The angular momentum $\mathrm{C} = \sum_j \mathbf{x}_j \wedge \mathbf{y}_j \in E \wedge E$ (a bivector, which identifies to a scalar when $d = 2$ and to a vector when $d = 3$)

The invariance by velocity shifts is associated with the first integral $\sum_j \left(m_j \mathbf{x}_j - \mathbf{y}_j t\right)$, which depends on time. But let us stick to autonomous vector fields and integrals. This invariance has the additional consequence that the dynamics does not depend on the fixed value of the linear momentum (put differently, this value can be adjusted arbitrarily by switching to an arbitrary inertial frame of reference), whereas the dynamics does depend on the fixed value of the angular momentum (see the paragraph on the reduction of Lagrange): for example, as a lemma of Sundman will show in Section 7, total collision may occur only if the angular momentum is zero. In the sequel, we will assume that the linear momentum is equal to zero whenever needed.

In addition to these Galilean symmetries, there is a much more specific *scaling invariance* due to the fact that the kinetic energy $|\mathbf{y}|^2/2$ and the force function $U(\mathbf{x})$ are homogeneous of respective degrees 2 and -1: if $\mathbf{x}(t)$ is a solution, so is $\mathbf{x}_\lambda(t) = \lambda^{-2/3}\mathbf{x}(\lambda t)$ for any $\lambda > 0$.

2.1. Reduction of the Problem by Translations and Isometries

The invariance by translations and isometries can be used to reduce the number of dimensions of the N-body problem. The first complete reduction of the three-body problem was carried out by Lagrange. Albouy-Chenciner generalized it for N bodies in $\mathbb{R}^d$, which we now outline. This reduction has proved efficient in particular in the study of relative equilibria or, recently, for numerical integrators.

But before fleshing out this construction, let us mention that a somewhat less elegant reduction, known as the "reduction of the node", was later obtained by Jacobi for three bodies, generalized by Boigey for four bodies and Deprit for an arbitrary number N of bodies. Jacobi's reduction has the disadvantage of breaking the symmetry between the bodies and of being rather specific (at least in its usual form) to the three-dimensional physical space. Yet it has proved more convenient in perturbative problems. Using this reduction, Chierchia-Pinzari managed to show that the planetary system is non-degenerate in the sense of Kolmogorov at the elliptic secular singularity (see *The Planetary N-Body Problem*).

Recall that $E = \mathbb{R}^d$ is the Euclidean vector space where motion takes place. Thus the state space (combined positions and velocities of the N bodies) is $\left(E^N\right)^2 = \{(\mathbf{x}, \dot{\mathbf{x}})\}$. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_N, \dot{\mathbf{e}}_1, \ldots, \dot{\mathbf{e}}_N)$ be the canonical basis of $\mathbb{R}^{2N}$. The map

$$\left(E^N\right)^2 \to E \otimes \mathbb{R}^{2N}, \quad (\mathbf{x}, \dot{\mathbf{x}}) \mapsto \boldsymbol{\xi} = \sum_{1 \le i \le N} \left(\mathbf{x}_i \otimes \mathbf{e}_i + \dot{\mathbf{x}}_i \otimes \dot{\mathbf{e}}_i\right) \tag{3}$$

is an isomorphism, which allows us to identify a state $(\mathbf{x}, \dot{\mathbf{x}})$ to the tensor $\boldsymbol{\xi}$.

The space E acts diagonally by translations on positions:

$$\mathbf{x} + \delta\mathbf{x}_0 = \left(\mathbf{x}_1 + \delta\mathbf{x}_0, \ldots, \mathbf{x}_N + \delta\mathbf{x}_0\right), \qquad \delta\mathbf{x}_0 \in E$$

and similarly (but separately: Newton's equations are invariant by separate translations on positions and on velocities) on velocities. The isomorphism above induces an isomorphism

$$\left(E^N/E\right)^2 \to E \otimes \mathcal{D}^2 \tag{4}$$

where $\mathcal{D}$ is what Albouy-Chenciner call the *disposition space* $\mathbb{R}^N/Vect(1,\ldots1)$. The space $E \otimes \mathcal{D}^2$ represents states up to translations, which we will still denote by the letter $\boldsymbol{\xi}$.

Let $\boldsymbol{\epsilon}$ denote the Euclidean structure of E. Pulled-back by $\boldsymbol{\xi}$, it becomes a symmetric tensor

$$\boldsymbol{\sigma} = \boldsymbol{\xi}^{\mathrm{T}} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{\xi} \in \left(\mathcal{D}^2\right)^{\otimes 2}, \tag{5}$$

which characterizes $\boldsymbol{\xi}$ up to the isometry $\boldsymbol{\iota} = \boldsymbol{\xi} \cdot \boldsymbol{\sigma}^{-1/2}$ of E (otherwise said, $\boldsymbol{\xi} = \boldsymbol{\iota} \cdot \boldsymbol{\sigma}^{1/2}$ is the standard polar decomposition). Hence the space $\left(\mathcal{D}^2\right)^{\otimes 2}$ represents states up to translations and isometries, called *relative states*.

For the sake of concreteness, write

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\beta} & \boldsymbol{\gamma} - \boldsymbol{\rho} \\ \boldsymbol{\gamma} + \boldsymbol{\rho} & \boldsymbol{\delta} \end{pmatrix}$$

the block decomposition of $\boldsymbol{\sigma}$, where $\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\rho} \in \mathcal{D}^{\otimes 2}$ and $\boldsymbol{\gamma}^{\mathrm{T}} = \boldsymbol{\gamma}$ and $\boldsymbol{\rho}^{\mathrm{T}} = -\boldsymbol{\rho}$. The space $\mathcal{D}^* = \left\{\mathbf{v}^* \in \mathbb{R}^{N*}, \mathbf{v}^* \cdot (1, \ldots, 1) = 0\right\}$ having no canonical basis, consider instead the generating family of covectors $\mathbf{e}_{ij}^* = \mathbf{e}_j^* - \mathbf{e}_i^*$ and $\dot{\mathbf{e}}_{ij}^* = \dot{\mathbf{e}}_j - \dot{\mathbf{e}}_i$ in $\mathcal{D}^{2*}$, where $\left(\mathbf{e}_1^*, \ldots, \mathbf{e}_N^*, \dot{\mathbf{e}}_1^*, \ldots, \dot{\mathbf{e}}_N^*\right)$ is the canonical basis of $\mathbb{R}^{2N*}$, and

$$\boldsymbol{\xi} \cdot \mathbf{e}_{ij}^* = \mathbf{x}_j - \mathbf{x}_i \quad \text{and} \quad \boldsymbol{\xi} \cdot \dot{\mathbf{e}}_{ij}^* = \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_i .$$

The blocks $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ being symmetric, they are determined by the identities

$$\begin{cases} \boldsymbol{\beta} \cdot \left(\mathbf{e}_{ij}^* \otimes \mathbf{e}_{ij}^*\right) = \left\|\mathbf{x}_j - \mathbf{x}_i\right\|^2 \\ \boldsymbol{\delta} \cdot \left(\dot{\mathbf{e}}_{ij}^* \otimes \dot{\mathbf{e}}_{ij}^*\right) = \left\|\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_i\right\|^2 \\ \boldsymbol{\gamma} \cdot \left(\mathbf{e}_{ij}^* \otimes \dot{\mathbf{e}}_{ij}^*\right) = \left(\mathbf{x}_j - \mathbf{x}_i\right) \cdot \left(\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_i\right), \end{cases} \tag{6}$$

involving only scalar products of mutual distances and velocities.

But what does the equation of dynamics become in this framework? The bilinear form on $\mathbb{R}^N$

$$\sum_{1 \le i \le N} m_i \left(\mathbf{e}_i^* - \mathbf{e}_{\mathrm{G}}^*\right) \otimes \mathbf{e}_i^*, \qquad \mathbf{e}_{\mathrm{G}}^* = \frac{1}{M}\left(m_1 \mathbf{e}_1^* + \cdots + m_N \mathbf{e}_N^*\right), \tag{7}$$

with $M = m_1 + \cdots + m_N$, descends to the quotient by $(1,\ldots,1)$ and induces the *mass scalar product* $\boldsymbol{\mu}$ on $\mathcal{D}$. Newton's equation then reads, in $E^* \otimes \mathcal{D}^*$,

$$\epsilon \cdot \ddot{\mathbf{x}} \cdot \boldsymbol{\mu} = dU, \qquad U = \sum_{i<j} \frac{m_i m_j}{\left\|\mathbf{x}_i - \mathbf{x}_j\right\|}, \tag{8}$$

provided $\mathbf{x}$ is thought of as an element of $E \otimes \mathcal{D}$ —an absolute configuration. The force function factorizes through relative positions: $U(\mathbf{x}) = \hat{U}(\boldsymbol{\beta})$, for it depends only on mutual distances. Since $d\hat{U}$ is a linear form on the space of symmetric tensors of $\mathcal{D}^{\otimes 2}$, it is itself symmetric. Hence,

$$dU \cdot \mathbf{x}' = d\hat{U} \cdot \left(\mathbf{x}'^{\mathrm{T}} \cdot \epsilon \cdot \mathbf{x} + \mathbf{x}^{\mathrm{T}} \cdot \epsilon \cdot \mathbf{x}'\right) = 2\epsilon \cdot \mathbf{x} \cdot d\hat{U} \cdot \mathbf{x}',$$

and the equation becomes

$$\ddot{\mathbf{x}} = 2\mathbf{x} \cdot \mathbf{A}, \tag{9}$$

provided we define the *Conley-Wintner endomorphism* of $\mathcal{D}^*$ as $\mathbf{A} = d\hat{U} \cdot \boldsymbol{\mu}^{-1}$. It is then straightforward to deduce the reduced equation:

$$\begin{cases} \dot{\boldsymbol{\beta}} = 2\boldsymbol{\gamma} \\ \dot{\boldsymbol{\gamma}} = \left(\mathbf{A}^{\mathrm{T}} \cdot \boldsymbol{\beta} + \boldsymbol{\beta} \cdot \mathbf{A}\right) + \boldsymbol{\delta} \\ \dot{\boldsymbol{\delta}} = 2\left(\mathbf{A}^{\mathrm{T}} \cdot \boldsymbol{\gamma} + \boldsymbol{\gamma} \cdot \mathbf{A}\right) - 2\left(\mathbf{A}^{\mathrm{T}} \dot{\boldsymbol{\rho}} - \boldsymbol{\rho} \cdot \mathbf{A}\right) \\ \dot{\boldsymbol{\rho}} = \mathbf{A}^{\mathrm{T}} \cdot \boldsymbol{\beta} - \boldsymbol{\beta} \cdot \mathbf{A}. \end{cases} \tag{10}$$

We have already defined the energy as

$$H = \frac{\|\mathbf{y}\|^2}{2} - U, \tag{11}$$

which induces a function on the phase space $\mathcal{P} = (E \otimes \mathcal{D}) \oplus \left(E \otimes \mathcal{D}^*\right)$, whose first term corresponds to absolute position $\mathbf{x}$ (modulo translations) and second term corresponds to absolute linear momentum $\mathbf{y}$ (acting on absolute velocities, modulo translations). Let $\boldsymbol{\omega}$ be the natural symplectic form on (the tangent space of) $\mathcal{P}$:

$$\boldsymbol{\omega} \cdot (\mathbf{x}, \mathbf{y}) \otimes (\mathbf{x}', \mathbf{y}') = \epsilon \cdot \left(\mathbf{x} \cdot y'^{\mathrm{T}} - \mathbf{x}' \cdot \mathbf{y}^{\mathrm{T}}\right). \tag{12}$$

The vector field $\mathbf{X} = (\dot{\mathbf{x}}, \dot{\mathbf{y}})$ associated with Newton's equation in $\mathcal{P}$ is nothing else than the Hamiltonian vector field of H with respect to $\boldsymbol{\omega}$: $i_{\mathbf{X}}\boldsymbol{\omega} = dH$. The inverse of $\boldsymbol{\omega}$

(as an isomorphism $\mathcal{P} \to \mathcal{P}^*$) is a Poisson structure $\boldsymbol{\pi}$, which can be pulled back by the transpose of the quotient by isometries, to a degenerate Poisson structure $\bar{\boldsymbol{\pi}} \in \mathcal{D}^{\otimes 2}$. The symplectic leaves of $\bar{\boldsymbol{\pi}}$ are the submanifolds obtained by fixing the rank of $\boldsymbol{\sigma}$ and the conjugacy invariants of the endomorphism $\boldsymbol{\omega}_{\mathcal{D}} \cdot \boldsymbol{\sigma}$ (the invariants of the angular momentum), where $\boldsymbol{\omega}_\mu$ stands for the symplectic form

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} \cdot (\mathbf{u}, \mathbf{v}) \otimes (\mathbf{u}', \mathbf{v}') = \boldsymbol{\mu} \cdot (\mathbf{u} \otimes \mathbf{v}' - \mathbf{u}' \otimes \mathbf{v}) \qquad (13)$$

of $\mathcal{D}^2$.

(N.B.: In this section where linear algebra has played an important role, we have written tensors in bold letters. But we will not consistently do so in the sequel, because vector spaces will be thought of as manifolds more often than not.)

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Biographical Sketch

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