ALGEBRA

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Contents

Summary

The six chapters *"Matrices, Vectors, Determinants, and Linear Algebra* ", "*Groups and Applications*", *"Rings and Modules"*, *"Fields and Algebraic Equations"*, *"Number Theory and Applications*", and "*Algebraic Geometry and Applications"* cover basic concepts and notions in algebra, which are usually taught to mathematics majors in colleges and universities at undergraduate and beginning graduate levels.

However, there are many other key concepts and notions in algebra that play important roles not only in mathematics but in many other disciplines in science thanks to their very abstraction.

Due to space limitation, only a few of them are explained here. The concepts and results in the six chapters are freely used.

1. Equivalence Relations

One of the basic concepts in modern mathematics is an equivalence relation on a set, which generalizes the notion of equality and which in fact appeared already in the six chapters.

An *equivalence relation* ~ is said to be defined on a set X if any given pair of elements

$x, y \in X$ satisfy either $x \sim y$ or $x \nsim y$ and if the following conditions are satisfied:

(Reflexive) $x \sim x$ holds for any $x \in X$.
(Symmetric) $x \sim y$ implies $y \sim x$.
(Transitive) $x \sim y$ and $y \sim z$ imply $x \sim z$.

More generally, a relation on a set X could be defined as a nonempty subset $R \subset X \times X$ of the product set of X with itself. In the case of an equivalence relation, the subset R is

$$R := \{(x, y) \in X \times X \mid x \sim y\},$$

which satisfies the following conditions corresponding to those above:

(Reflexive) $(x, x) \in R$ for any $x \in X$.
(Symmetric) $(x, y) \in R$ implies $(y, x) \in R$.
(Transitive) $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$.

The equality $=$ of elements in a set X is certainly an equivalence relation, with $R = \{(x, x) \mid x \in X\}$ being the diagonal.

To define an equivalence relation on X also coincides with giving a *decomposition of* X *into a disjoint union of nonempty subsets*

$$X = \bigcup_{i \in I} X_i \quad \text{(disjoint)} \qquad \text{with } X_i \cap X_j = \emptyset \ (i \neq j)$$

The corresponding equivalence relation is

$$x \sim y \quad \text{if and only if} \quad x, y \in X_i \text{ for some } i \in I.$$

Given an equivalence relation ~ on X , let

$$X(a) := \{x \in X \mid x \sim a\} \quad \text{for each } a \in X,$$

which is called the equivalence class containing a . Then for any $a, b \in X$ one obviously has either $X(a) = X(b)$ or $X(a) \cap X(b) = \emptyset$. One can choose an element from each equivalence class to form a subset of *representatives* $\{a_i\}_{i \in I} \subset X$ giving rise to the decomposition

$$X = \bigcup_{i \in I} X(a_i) \quad \text{(disjoint)}$$

corresponding to the equivalence relation.

Another way of defining an equivalence relation is to give an *onto* map $\pi : X \to Q$ called the *projection* to the *quotient set* Q. The corresponding equivalence relation is $x \sim y$ if and only if $\pi(x) = \pi(y)$. This gives rise to the corresponding decomposition

$$X = \bigcup_{q \in Q} \pi^{-1}(q)$$

Given an equivalence relation ~ on X, the corresponding quotient set $Q =: X/\sim$ is nothing but the set of equivalence classes, and the projection $\pi : X \to X/\sim$ sends each $a \in X$ to its equivalence class $\pi(a) := X(a)$.

Here are examples of equivalence relations:

(1) Let H be a subgroup of a group G. For $x, y \in G$, define $x \equiv y \left(\operatorname{mod} H\right)$ by $x^{-1}y \in H$, which is obviously equivalent to $xH = yH$. This is easily seen to be an equivalence relation, and the equivalence class containing x is the left coset xH. The corresponding quotient map is $\pi : G \to G/H$ sending $x \in G$ to $\pi(x) := xH$. There is an obvious analog with respect to the right cosets Hx.
(2) When N is a normal subgroup of a group G, one has $xN = Nx$ for each $x \in G$. In this case G/N is a group under the group law $(xN)\cdot(yN) := xyN$ that turns out to be well-defined thanks to the normality of N in G. One calls G/N the quotient group with respect to N. The projection $\pi : G \to G/N$ is an onto homomorphism of groups with kernel $\ker(\pi) = N$.
(3) Let R be a commutative ring with $1 \neq 0$, and $I \subset R$ an ideal with $I \neq R$. For $x, y \in R$, define $x \equiv y \left(\operatorname{mod} I\right)$ by $x - y \in I$. This is easily seen to be an equivalence relation. The equivalence class containing $x \in R$ is the coset $x + I$. The quotient set R/I turns out to be a commutative ring with respect to the addition $(x+I)+(y+I) := x+y+I$ and the multiplication $(x+I)\cdot(y+I) := xy+I$ with the identity $1+I \neq 0+I$. One calls R/I the residue ring of R with respect to I. The corresponding projection $\pi : R \to R/I$ sending $x \in R$ to $\pi(x) := x+I$ is an onto ring homomorphism with kernel $\ker(\pi) = I$.
(4) For a commutative ring R with $1 \neq 0$, let M be an R-module, and $N \subset M$ an R-submodule. For $x, y \in M$, define $x \equiv y \left(\operatorname{mod} N\right)$ by $x - y \in N$. This is easily seen to be an equivalence relation. The equivalence class containing $x \in M$ is the coset $x + N$. The quotient set M/N is an R-module with respect to the addition $(x+N)+(y+N) := x+y+N$ and the scalar multiplication $a(x+N) := ax+N$ for $a \in R$. One calls M/N the quotient R-module of M with respect to N. The corresponding projection $\pi : M \to M/N$ sending $x \in M$ to $\pi(x) := x+N$ is an onto homomorphism of R-modules with kernel $\ker(\pi) = N$.
($4'$) For elements x, y in a vector space V over a field K (e.g., $K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$) and a K-linear subspace $W \subset V$, define $x \equiv y \left(\operatorname{mod} W\right)$ by $x - y \in W$. This is easily seen to be an equivalence relation. The equivalence class containing $x \in V$ is the coset $x + W$. The quotient set V/W is a K-vector space with respect to the addition

$(x+W)+(y+W):=x+y+W$ and the scalar multiplication $a(x+W):=ax+W$ for $a\in K$. One calls V/W the quotient K-vector space of V with respect to W. The corresponding projection $\pi:V\to V/W$ sending $x\in V$ to $\pi(x):=x+W$ is an onto K-linear map with kernel $\ker(\pi)=W$.

(5) A group G is said to *act* on a set X if there is given a map $G\times X\to X$ sending $(g,x)\in G\times X$ to $gx\in X$ that satisfies

$$g(g'x)=(gg')x, \text{ and } ex=x \qquad \text{for all } g,g'\in G, x\in X.$$

$x,y\in X$ are defined to be equivalent if they are in the same G-orbit, that is, there exists $g\in G$ such that $y=gx$. This is easily seen to be an equivalence relation. The equivalence class containing x is its G-orbit $Gx:=\{gx\mid g\in G\}$. The corresponding decomposition is the orbit decomposition

$$X=\bigcup_{i\in I}Gx_i \quad \text{(disjoint)}.$$

The quotient set consisting of G-orbits is called the orbit space and is denoted X/G (one also denoted this by $G\backslash X$ to emphasize that the action is from the left) with the projection map $\pi:X\to X/G$ sending $x\in X$ to its G-orbit $\pi(x)=Gx$.

(6) In the $(n+1)$-dimensional complex vector space $\mathbb{C}^{n+1}$, let $X:=\mathbb{C}^{n+1}\setminus\{O\}$ with $O:=(0,0,\ldots,0)\in\mathbb{C}^{n+1}$. The multiplicative group $\mathbb{C}^\times:=\mathbb{C}\setminus\{0\}$ of nonzero complex numbers acts on X by scalar multiplication

$$\lambda(t_0,t_1,\ldots,t_n):=(\lambda t_0,\lambda t_1,\ldots,\lambda t_n)$$

for $\lambda\in\mathbb{C}^\times$ and $(t_0,t_1,\ldots,t_n)\in X$. The $\mathbb{C}^\times$-orbit of $(t_0,t_1,\ldots,t_n)$ is

$$[t_0,t_1,\ldots,t_n]:=\{(\lambda t_0,\lambda t_1,\ldots,\lambda t_n)\mid\lambda\in\mathbb{C}^\times\}.$$

The orbit space is the n-dimensional complex projective space

$$\mathbb{P}^n:=(\mathbb{C}^{n+1}\setminus\{O\})/\mathbb{C}^\times.$$

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Biographical Sketch

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