TOPOLOGY

Takashi Tsuboi
Graduate School of Mathematical Sciences, the University of Tokyo, Japan

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Contents

Summary

Topology is a branch of mathematics which studies the shape of spaces and the configurations of a space mapped in another space. The current theory of topology consists of “general topology” which is the foundation to talk about convergence in various spaces and continuity of maps between spaces, “homotopy theory and homology theory” which define various invariants to distinguish spaces and configurations, and “applications” of these general results to the interesting objects such as manifolds.

1. Introduction

Topology is a branch of mathematics which studies the shape of spaces and the configurations of a space mapped in another space.

One of the origins of topology theory is usually attributed to Euler's argument to deduce the impossibility to find a route in Königsberg passing each of 7 bridges once and only once. The configuration of the bridges of Königsberg is represented by a graph with 4 vertices and 7 edges, the question is the existence of a path passing through each edge once and only once. Euler looked at the number of attached edges at each vertex, and found that for any connected graph with at most 2 vertices with odd number of attached edges, it is possible to find such a path and otherwise it is impossible to find such a path. In the graph representing Königsberg, the 4 vertices have 5, 3, 3 and 3 attached edges, respectively, and hence it is impossible to find the desired route.
Euler also found that for a convex polyhedron,

$$(\textit{the number of vertices}) - (\textit{the number of edges}) + (\textit{the number of faces}) = 2.$$

The number of left-hand-side does not depend on the polyhedral arrangements and it is called the Euler-Poincaré characteristic number. This is still the most important invariant to distinguish the shapes of polyhedra.

Another origin of topology is the study of the convergences of sequences and series such as Fourier series. It took rather long time to understand that the pointwise limit of continuous functions may not be a continuous function while the uniform convergent limit of continuous functions is continuous. After the set theory was established, the topology is thought as a structure on a set and the theory of topology gives a basis to discuss convergence as well as the continuous deformations of various objects.

The current theory of topology consists of "general topology" which is the foundation to talk about convergence in various spaces and continuity of maps between spaces, "homotopy theory and homology theory" which define various invariants to distinguish spaces and configurations, and "applications" of these general results to the interesting objects such as manifolds.

2. Convergence of Sequences, Continuity of Maps, General Topology

2.1. Metric Spaces and the Convergence of Sequences

First we look at the spaces where there is a notion of distance.

In the n-dimensional Euclidean space $\mathbb{R}^n$, the distance between two points $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ is given by $d(x,y)=\|x-y\|=\sqrt{\sum_{k=1}^{n}(x_k-y_k)^2}$. This distance function $d:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ satisfies the following properties.

1) For x, $y\in X$, $d(x,y)=d(y,x)$;

2) For x, $y \in X$, $d(x,y) \geqq 0$ and $d(x,y) = 0$ if and only if $x = y$;
3) For x, y, $z \in X$, $d(x,z) \leqq d(x,y) + d(y,z)$.

By definition, a *metric space* is a set X with a real-valued function $d : X \times X \to \mathbb{R}$ satisfying the precedent properties (1)–3). There are many interesting examples of metric spaces, e.g., the space $B(S)$ of real-valued bounded functions on a set S is a metric space by defining $d(f,g) = \sup_{x \in S} d(f(x), g(x))$ for f, $g \in B(S)$.

For a metric space X, we can define the ε neighborhood of a point $x \in X$ and what is the convergent sequence. For a positive real number ε, the ε (open) *neighborhood* $B_\varepsilon(x)$ of $x \in X$ is defined by $B_\varepsilon(x) = \{y \in X \mid d(y,x) < \varepsilon\}$. A *sequence* (of points) in a metric space X is a mapping from $\mathbb{N}$ to X. It is usually written as $\{x_i\}_{i \in \mathbb{N}}$. A sequence $\{x_i\}_{i \in \mathbb{N}}$ in a metric space X ($x_i \in X$) is said to *converge* to $x \in X$ if for any $\varepsilon > 0$, $\{i \in \mathbb{N} \mid x_i \notin B_\varepsilon(x)\}$ is a finite set. In this case, x is called the *limit* of the sequence and is written as $\lim_{i \to \infty} x_i = x$. If a sequence $\{x_i\}_{i \in \mathbb{N}}$ in a metric space converges, it converges to a unique point in the metric space X.

A subset A of a metric space X is said to be *closed* if any sequence $\{x_i\}_{i \in \mathbb{N}}$ of points in A converges to a point $x \in X$, then $x \in A$.

A sequence $\{x_i\}_{i \in \mathbb{N}}$ in a metric space X is called a *Cauchy sequence* if for any $\varepsilon > 0$, there is a number k such that $d(x_i, x_j) < \varepsilon$ for i, $j \geqq k$. A convergent sequence is a Cauchy sequence. But a Cauchy sequence may not converge in general. For, one may think of the sequence $\{1/n\}$ on the open half line $\{x \in \mathbb{R} \mid x > 0\}$. If all Cauchy sequences converge, the metric space is called *complete*. The n-dimensional Euclidean space is complete and a closed subset A of a complete metric space is complete with respect to the distance function restricted to A.

A map $f : X \to Y$ between metric spaces is *continuous* if for any $x \in X$ and a positive real number ε, there exists a positive real number δ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. This is equivalent to say that if a sequence $\{x_i\}_{i \in \mathbb{N}}$ converges to x, then the sequence $\{f(x_i)\}_{i \in \mathbb{N}}$ converges to $f(x)$.

An *open set* U of a metric space X is defined to be a set U such that if $x \in U$ there is a positive real number ε satisfying $B_\varepsilon(x) \subset U$. That U is open is equivalent to that the complement $X \setminus U$ is closed. By using the notion of open sets, a map $f : X \to Y$ between metric spaces is continuous if for any open set U of Y, $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$ is an open set of X.

2.2. Abstract Topology on Sets

We can think of the continuity of the maps between spaces which are not metric spaces. First we notice that the set $\mathcal{O}$ of open sets defined in a metric space X has the following properties:

1) $X \in \mathcal{O}$ and $\varnothing \in \mathcal{O}$;
2) For a finite subset $\{O_i\}_{i=1,\ldots,k} \subset \mathcal{O}$, $\bigcap_{i=1}^{k} O_i \in \mathcal{O}$;
3) For any subset $\{O_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{O}$, $\bigcup_{\lambda\in\Lambda} O_\lambda \in \mathcal{O}$.

Now for any set X, we define a *topology* of X to be the set $\mathcal{O}$ of subsets of X satisfying the above three properties (1)–3). Elements of $\mathcal{O}$ are called *open sets*. The set X with a topology is called a *topological space* or simply a *space*.

Using the notion of topology, we can define the notion of continuity of a map between topological spaces. Given the set $\mathcal{O}_X$ of open sets of X and the set $\mathcal{O}_Y$ of open sets of Y, a map $f: X \to Y$ is *continuous* if $U \in \mathcal{O}_Y$ implies $f^{-1}(U) \in \mathcal{O}_X$.

When the set $\mathcal{O}$ of open sets of X is given, the set N_x of *open neighborhoods* of $x \in X$ is defined as $N_x = \{O \in \mathcal{O} \mid x \in O\}$. Given the sets of open neighborhoods N_x^X of $x \in X$ and N_y^Y of $y \in Y$, $f: X \to Y$ is *continuous at* x_0 if for any open neighborhood $V \in N_{f(x_0)}^Y$ there is an open neighborhood $U \in N_{x_0}^X$ such that $f(U) \subset V$.

This abstraction is useful to understand various notions concerning convergence and the continuity.

A *closed set* is the complement of an open set. Let $\mathcal{C}$ be the set of closed sets of X. For a subset S of X, the *closure* $\overline{S}$ of S is the smallest closed set containing S, that is, $\overline{S} = \bigcap_{C\in\mathcal{C}, S\subset C} C$. For a metric space X, the closure of a subset S consists of all the points which are the limits of sequences in S which are convergent in X.

For any subset A of a topological space X, A has the *induced topology* defined by the set of open sets given by $\{O \cap A \mid O \in \mathcal{O}\}$. With this topology, A is called a (topological) *subspace* of X. The inclusion map of any subspace A to X is a continuous map.

For two topological spaces X and Y, the direct product $X \times Y$ which is the set of the ordered pairs (x, y) where $x \in X$ and $y \in Y$ is a topological space. The open sets are the subsets of $X \times Y$ which is written as a union $\bigcup_{\lambda\in\Lambda} U_\lambda \times V_\lambda$ of direct products of open sets $U_\lambda \in \mathcal{O}_X$ and $V_\lambda \in \mathcal{O}_Y$ $(\lambda \in \Lambda)$.

A map $h: X \to Y$ between topological spaces is called a *homeomorphism* if h is a bijection and $h: X \to Y$ and its inverse map $h^{-1}: Y \to X$ are continuous. If there is a homeomorphism $h: X \to Y$, X and Y are said to be *homeomorphic*. For example, the open ball $\{x \in \mathbb{R}^n \mid d(0, x) < 1\}$ in the n-dimensional Euclidean space $\mathbb{R}^n$ is homeomorphic to the n-dimensional Euclidean space $\mathbb{R}^n$.

The principal problem in topology theory is to determine whether two topological spaces are homeomorphic or not. To show that they are homeomorphic it is necessary to show the way to construct a homeomorphism between them. To show that they are not homeomorphic it is necessary to define an invariant which has the same value if they are homeomorphic and to show that the values of the invariant are different for the two spaces. Homotopy groups and homology groups explained later are topological invariants and used to show that two topological spaces are different.

2.3. Separation Axioms and Countability Axioms

The topological spaces usually treated have several nice properties which the metric spaces satisfy. Such properties are also formulated in an abstract way.

A topological space X is called a *Hausdorff space*, if for any two points x and y of X, there are open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Metric spaces are Hausdorff spaces. In a Hausdorff space one-point set $\{x\}$ is a closed set as is the case in metric spaces.

A Hausdorff space X is called a *normal space*, if for any two disjoint closed subsets C_0 and C_1 of X, there are open sets U_0 and U_1 such that $C_0 \subset U_0$, $C_1 \subset U_1$ and $U_0 \cap U_1 = \emptyset$. In a normal space X, for any two disjoint closed subsets C_0 and C_1 of X, there is a continuous function $f : X \to [0,1]$ such that $C_0 = f^{-1}(0)$ and $C_1 = f^{-1}(1)$. This statement is called Urysohn's Lemma. Thus for a normal space, there are a lot of real-valued continuous functions on X and one can distinguish closed sets by them.

A subset B_x of the set N_x of open neighborhoods of x is called a *base* of N_x if for any element U of N_x there is $V \in B_x$ such that $V \subset U$. If there is a countable base B_x of N_x at every point $x \in X$, the topological space X is said to satisfy the *first axiom of countability*. If X is a metric space, the topology of X satisfies the first axiom of countability.

A subset B of $\mathcal{O}$ is a *base* of $\mathcal{O}$ if any element U of $\mathcal{O}$ is written as $U = \bigcup_{V \in B, V \subset U} V$. If there is a countable base B of $\mathcal{O}$, X is said to satisfy the *second axiom of countability*. If X satisfies the second axiom of countability, then it of course satisfies the first axiom of countability.

For the direct product $X \times Y$ of topological spaces X and Y, $\{U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$ is the base of $\mathcal{O}_{X \times Y}$. If X and Y satisfy the second axiom of countability, then $X \times Y$ satisfies it.

The n-dimensional Euclidean space $\mathbb{R}^n$ is a metric space, hence it satisfies the first axiom of countability. $\mathbb{R}^n$ also satisfies the second axiom of countability. It is easy to see that any subspace of $\mathbb{R}^n$ with the induced metric satisfies both axioms of countability.

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Biographical Sketch

Takashi Tsuboi, born 1953 in Hiroshima, Japan
Education:

BS in Mathematics, the University of Tokyo, Japan (March, 1976).
MS in Mathematics, the University of Tokyo, Japan (March, 1978).
Ph.D. in Mathematics, the University of Tokyo, Japan (February, 1983).
Positions held:
Assistant, Department of Mathematics, the University of Tokyo, Japan (April, 1978--March, 1985)
Assistant Professor, Department of Mathematics, the University of Tokyo, Japan (April, 1985--September, 1993)
Professor, Graduate School of Mathematical Sciences, the University of Tokyo, Japan (October, 1993 to date)