INFINITE ANALYSIS

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Keywords: exactly solvable models, the Ising model, soliton equation, conformal field theory, the XXZ model, the sine-Gordon model, free Fermions, free Bosons, vertex operator, affine Lie algebra, quantum affine algebra

Contents

Summary

Solvable models are discussed in connection with the representation theory of infinite dimensional algebras. Vertex operators, which realize representations and intertwine

different representations, give mathematical pictures to physical models in statistical mechanics and quantum field theory.

1. Introduction

In this article we give a survey of solvable models in statistical mechanics and quantum field theory. Problems in these fields of physics possess a peculiar feature. Physical systems in consideration have infinite degrees of freedom. For example, a magnet consists of a huge number of iron atoms. Each atom bears one degree of freedom.

Mathematically, degrees of freedom appear as variables over which summations or integrations are taken. Because of such a large number of variables, it is not easy to obtain analytical solutions in general. However, in reality, there are plenty of interesting models that are exactly solvable. In this article, we take five different topics in solvable models, and give a brief account of known results in the case of each model. We consider the two-dimensional Ising model, soliton equations, the $SU(2)$ WZNW model in conformal field theory, the XXZ spin chain, and the sine-Gordon model.

A common feature of these models is that the systems have a large set of symmetries. In the limit where the degrees of freedom become infinite, the systems acquire infinite dimensional symmetries. Mathematically, this is understood as the situation where the systems can be identified with infinite dimensional representations of infinite dimensional algebras. Representation theory of infinite dimensional algebras such as the free Fermion algebra, the free Boson algebra, the affine Lie algebras, the Virasoro algebra and the quantum affine algebras plays essential roles in obtaining analytical solutions of the models. At the same time, solvable models in physics provide us with new problems and new methods in representation theory. We call the field of interactions between solvable models in physics and representation theory of infinite dimensional algebras INFINITE ANALYSIS.

In infinite analysis, a key ingredient is the existence of the spectral variable. Operators depending on the spectral variable are called vertex operators. Physically, the spectral variable appears as a space (or momentum) variable. Therefore, vertex operators are nothing but a kind of field operators. However, it is essential that vertex operators in solvable models describe the symmetries of the systems in a mathematically well-defined manner. They realize actions of infinite dimensional algebras and intertwine different representations. Properties of vertex operators are described in terms of their operator products. Functions of several complex variables appear as matrix elements of products of vertex operators. Vertex operators incarnate infinite dimensional symmetries as difference/differential equations for these functions. In this way, non-linear Painleve equations, KP and KdV equations, KZ and qKZ equation arise in infinite analysis.

2. Ising Model and Monodromy Preserving Deformation

We introduce the two-dimensional Ising model. It is a model in classical statistical mechanics on the two-dimensional lattice. It has a critical point where the physical quantities have singularities. Exact expressions for the free energy per site, the spontaneous magnetization, and the correlation functions are given. In the continuum limit, it gives a model of quantum field theory.

2.1. Two-dimensional Ising Model and Onsager's Result

In this section, we give the definition of the two-dimensional Ising model, and give Onsager's result for the free energy per site. It has a singularity when the parameters of the model satisfy an algebraic relation.

Consider fluctuation variables $\sigma_{i,j} = \pm 1$ where $i \in \mathbb{Z}/M\mathbb{Z}$ and $j \in \mathbb{Z}/N\mathbb{Z}$. We assume that *M*, *N* are even. A configuration *C* is a mapping

$$C : (\mathbb{Z}/M\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z}) \to \{\pm 1\}, \quad (i, j) \mapsto \sigma_{i,j}.$$

There are 2^{MN} configurations on the two-dimensional lattice $(\mathbb{Z}/M\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$ of size $M \times N$. An element (i, j) of $(\mathbb{Z}/M\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$ is called "site" or "vertex" of the lattice, and a pair of neighboring sites $(i, j), (i, j+1)$ or $(i, j), (i+1, j)$ is called "edge" of the lattice.

For each configuration *C* we define the energy $E(C)$ by

$$E(C) = -E_1 \sum_{i,j} \sigma_{i,j}\sigma_{i+1,j} - E_2 \sum_{i,j} \sigma_{i,j}\sigma_{i,j+1},$$

where $E_1, E_2 > 0$. This implies that the energy of a configuration is the sum of local energies at each edge.

We set $K_i = \frac{E_i}{kT}$. We use (K_1, K_2) as parameters of the model. Set $c_i = \cosh 2K_i$ and $s_i = \sinh 2K_i$.

The Boltzmann principle of classical statistical mechanics asserts that the relative probability of occurrence of a configuration *C* is given by

$$e^{-\frac{E(C)}{kT}},$$

where *k* is the Boltzmann constant and *T* is temperature. The quantity $e^{-\frac{E(C)}{kT}}$ is called the Boltzmann weight of the configuration *C*. The partition function is the following configuration sum.

$$Z_{M,N} = \sum_C e^{-\frac{E(C)}{kT}}.$$

The free energy per site is

$$f_{M,N} = -\frac{kT}{MN}\log Z_{M,N}.$$

We are interested in the thermo-dynamic limit $f = \lim_{M,N\to\infty} f_{M,N}.$ Onsager obtained

$$-\frac{f}{kT} = \log 2 + \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\log(c_1c_2 - s_1\cos\theta_1 - s_2\cos\theta_2)d\theta_1 d\theta_2.$$

We introduce the dual parameters (K_2^*, K_1^*) by

$$\sinh 2K_i \sinh 2K_i^* = 1. \qquad (2.1)$$

Onsager's result shows that the free energy has a singularity at the self dual point, i.e.,

$$\sinh 2K_1 = \sinh 2K_2^*. \qquad (2.2)$$

In terms of the temperature *T*, the corresponding point $T = T_{\mathrm{c}}$ is called the critical temperature.

2.2. Transfer Matrix

In this section, we define the transfer matrix. The calculation of the free energy per site reduces to the diagonalization of the transfer matrix.

We can write the partition function $Z_{M,N}$ as the trace of a matrix of size 2^{MN}. The matrix is called the transfer matrix. It is defined as follows. Consider the two-dimensional space $\mathbb{C}^2$ for representing a spin variable ± 1. We use the following Pauli matrices on this space.

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

We split the two-dimensional lattice into one-dimensional slices $\{m\}\times(\mathbb{Z}/N\mathbb{Z})$, and we consider the *N*-fold product space of the spin space (denoted by $(\mathbb{C}^2)^{\otimes N}$), representing the configuration space on the one-dimensional slice. We denote the operator σ^x, etc., acting on the *n*-th component of the tensor product by σ_n^x, etc… Set

$$\begin{aligned}(V_1)_N &= e^{K_1\sum_{n\in\mathbb{Z}/N\mathbb{Z}}\sigma_n^z\sigma_{n+1}^z},\\ (V_2)_N &= (2s_2)^{N/2}e^{K_2^*\sum_{n\in\mathbb{Z}/N\mathbb{Z}}\sigma_n^z}.\end{aligned}$$

Define the transfer matrix V_N by

$$V_N = (V_1)_N^{\frac{1}{2}} (V_2)_N (V_1)_N^{\frac{1}{2}}.$$

We have

$$Z_{M,N} = \mathrm{trace}_{(\mathbb{C}^2)^{\otimes N}} (V_N)^M.$$

Suppose that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_{2^N}$$

are the eigenvalues of V_N. We have

$$Z_{M,N} = \lambda_1^M + \lambda_2^M + \cdots + \lambda_{2^N}^M.$$

In the thermodynamic limit, we can ignore the second and the smaller eigenvalues, and we have

$$\lim_{M,N\to\infty} \frac{1}{MN} \log Z_{M,N} = \lim_{N\to\infty} \frac{1}{N} \log(\lambda_1)\big|_{M\to\infty}.$$

In this way, the calculation of the free energy reduces to the calculation of the largest eigenvalue of the transfer matrix. The diagonalization of the transfer matrix is the basic problem in the model.

2.3. Harmonic Oscillator

In this section, we give an algebraic method in diagonalization of the harmonic oscillator Hamiltonian. The Heisenberg algebra appears in the method. We will see in the subsequent sections that one can reduce various models to a system of infinitely many harmonic oscillators.

In general, in quantum physics, the basic problem is the diagonalization of the Hamiltonian. The simplest example of the diagonalization is the harmonic oscillator in quantum mechanics. The Hamiltonian is given by

$$H_{h.o.} = -\frac{1}{2}\left(\frac{d^2}{dx^2} - x^2\right).$$

A three dimensional Lie algebra plays the central role in the diagonalization. Set

$$P = \frac{d}{dx} + x,$$
$$Q = -\frac{1}{2}\left(\frac{d}{dx} - x\right).$$

We have the commutation relation

$$[P,Q] = 1. \quad (2.3)$$

The Lie algebra spanned by P, Q and 1 is called the Heisenberg Lie algebra. The Hamiltonian belongs to the universal enveloping algebra of the Heisenberg Lie algebra.

$$H_{h.o.} = QP + \frac{1}{2}.$$

We have the commutation relations

$$[H_{h.o.}, P] = -P, \quad [H_{h.o.}, Q] = Q.$$

If v is an eigenvector of $H_{h.o.}$, then Pv (resp., Qv) (unless it is zero) is also an eigenvector, and the eigenvalue decreases (resp., increases) by 1.

The eigenvector v_0 corresponding to the smallest eigenvalue is obtained by solving the equation $Pv_0 = 0.$ We have

$$v_0 = e^{-\frac{1}{2}x^2}.$$

The vector v_0 is called the vacuum state. It has the eigenvalue $\frac{1}{2}$ of the Hamiltonian. Other eigenvectors are constructed by the operator Q as $Q^n v_0$. They are called the excited states. The state $Q^n v_0$ can be considered as consisting of n particles. Each particle has the energy 1. There is no interaction energy of these particles in the sense that the total energy is the sum of each constituents (apart from the vacuum energy), and there is no limit of the number of particles. For this reason, the harmonic oscillator is "free". The operator P and Q are annihilation and creation operators of the model.

2.4. Clifford Algebra and Clifford Group

In this section, we introduce an algebraic structure called the Clifford algebra. It contains a group called the Clifford group, which underlies the solvability of the two-dimensional Ising model. Both the transfer matrix and the spin operators belong to the Clifford group.

The harmonic oscillator has only one degree of freedom. The physical states are represented in terms of functions of a single variable x. The two-dimensional Ising model has infinite degrees of freedom in the thermodynamic limit. The vector space on which the transfer matrix acts is the N-fold tensor product of $\mathbb{C}^2$. Each component of the tensor product represents one degree of freedom, and therefore, in the limit $N \to \infty$, the system has the infinite degrees of freedom.

When N is finite, the tensor product

$$\mathcal{F}_N = (\mathbb{C}^2)^{\otimes N}$$

can be viewed as representation space of the Clifford algebra $\mathrm{End}_{\mathbb{C}}(\mathcal{F}_N)$ generated by the Fermions $p_n, q_n (n = 1, \ldots, N)$

$$p_n = \sigma_1^x \cdots \sigma_{n-1}^x \sigma_n^z,$$
$$q_n = \sigma_1^x \cdots \sigma_{n-1}^x \sigma_n^y.$$

They satisfy the anti-commutation relations

$$[p_n, p_{n'}]_+ = [q_n, q_{n'}]_+ = 2\delta_{n,n'}, [p_n, q_{n'}]_+ = 0.$$

The transformation of the operators from the Pauli matrices to the Fermions are called the Jordan-Wigner transformation. Let

$$W_N = (\oplus_{n=1}^N \mathbb{C} p_n) \oplus (\oplus_{n=1}^N \mathbb{C} q_n) \simeq \mathbb{C}^{2N}$$

be the space of Fermions. We define the Clifford group G_N,

$$G_N = \left\{ g \in \mathrm{End}_{\mathbb{C}}(\mathcal{F}_N); g \text{ is invertible and } gW_N g^{-1} = W_N \right\}$$

The element $g \in G_N$ induces an orthogonal transformation T_g,

$$T_g(x) = gxg^{-1},$$

of W_N, in which the inner product is given by $\langle x, x' \rangle = [x, x']_+$. The element $g \in G_N$ is uniquely determined (up to a constant) by T_g. When $N \to \infty$, we have

$$\begin{aligned} \dim G_N &\ll \dim \mathrm{End}_{\mathbb{C}}(\mathcal{F}_N), \\ 2\infty &\ll 2^{\infty}. \end{aligned}$$

The key fact, which ensures the solvability of the two-dimensional Ising model, is that the transfer matrix "essentially" belongs to the Clifford group. Let $\mathcal{F}_N^{(\pm)}$ be the subspace of $\mathcal{F}_N$ consisting of even/odd numbers of Fermions. There exist group elements $V_N^{(\pm)}$ such that

$$V_N\big|_{\mathcal{F}_N^{(\pm)}} = V_N^{(\pm)}.$$

In this way, the diagonalization of the transfer matrix reduces to the diagonalization of $T_{V_N^{(\pm)}} \in O(W_N)$. We omit the details of the diagonalization for finite size. In the next section, we give the result in the thermodynamic limit.

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Biographical Sketch

Tetsuji Miwa was born on February 10, 1949. He received the BS and MS, degrees from University of Tokyo in 1971 and 73 respectively and DSc degree from Kyoto University in 1981.

Since 1973 he is with Kyoto University in various positions, 1973-1984 as Assistant Research, Institute for Mathematical Sciences (RIMS), 1984-1992 as Associate Professor RIMS, 1993-2000 as Professor RIMS . In April 2000 he took the position of Professor, Graduate School of Science, Kyoto University. Dr. Miwa received the Autumn Prize of Mathematical Society of Japan on Oct.3, 1987 and the Asahi Prize on Jan.27, 2000.