FOURIER ANALYSIS AND INTEGRAL TRANSFORMS

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Contents

Summary

This section presents some basic matters on the Fourier analysis and several integral transforms, which are fundamental for analyzing a function in terms of simple functions and synthesizing them. Section 1 treats the theory of Fourier series. The Fourier coefficients which are defined as integral transforms (Section 1.1) and many integrals which are connected with Fourier analysis are closely related to convolution (Section 1.2). The convergence (divergence) problem of Fourier series has a long history and several important results are described here (Section 1.3, Section 1.4). Several important results of Fourier series are developed with the aid of analytic functions in the disk (Section 1.5). In the same section a definition of Hardy space which consists of analytic functions is given and F. and M. Riesz theorem is mentioned. In Section 1.6 theorems on orthogonal expansions and some well known examples of orthonormal functions are given.

Wavelets are lucidly constructed from multiresolution analysis (Section 2). In this section only a first step to wavelet theory is given. For examples of wavelets and details, the reader may see bibliography.
Most parts of the theory of Fourier transform follow from Fourier series themselves. In Section 3 well known summability kernels on the real line such as the Fejér, Poisson and Gauss-Weierstrass kernels are introduced. The relation between differentiation and the Fourier transform is explained in the Schwartz space.

Fourier analysis is also developed on a locally compact Abelian group which is described in Section 4. In particular, a finite cyclic group case is picked up and applied to the algorithms for numerical computation of Fourier transforms (Section 5). Section 6 is devoted to integral transforms: Mellin, Hankel, Laplace and wavelet transforms are introduced and mainly the inversion formulas are treated.

1. Introduction

A brief history of Fourier analysis: The basic concept of Fourier analysis is to decompose a periodic function into simple harmonic functions, and synthesize them to produce a given function. These concepts appeared, as modern mathematics, in the middle of the 18th century.

D.Bernoulli(1700-82), d'Alembert(1717-83), Euler(1707-83), Lagrange(1736-1813), and Fourier (1768-1830) came up to the idea to represent a function by a sum of cosine and sine functions in their studies of the equation of vibrating strings.

The method of Fourier analysis is quite general in describing the solutions of the heat equation and the wave equation.

Initiated by problems in physics and engineering science, Fourier analysis developed under mutual influences of many other mathematical fields: real analysis, the Cantor set theory, complex analysis, theory of integral, theory of partial differential equations, probability theory, group theory, number theory, wavelet theory, and so on. It is closely related to the theory integrals and is now established legitimately based on Lebesgue integration, and the theory is regarded as one of the most fundamental fields of modern analysis with widespread applications in physics, technologies and statistics.

Wavelet: The notion of wavelet is simultaneously introduced at the beginning of the 1980s arising out of the needs of harmonic analysis and engineering, and nowadays it plays an important role in these fields. In Section 2 only the definition is given. For detail refer Daubechies[?], Meyer[?] or Hernández and Weiss[?]

Integral transforms: The concept of an integral transform originated from the Fourier integral formula. The theory of Laplace transform is intimately connected with methods of solving differential equations. Its method is particularly useful for finding solutions of initial value problems in differential equations. In the last section some of typical integral transforms are mentioned.

For the formulae of integral transform, see Erdéli[?], Erdéli, Magnus, Oberhettinger, and Tricomi[?], or Gradshteyn and Ryzhik[?]

2. Fourier Series

2.1. Definition

Let $\mathbb{T}$ be the unit interval $[0,1) = \{x : 0 \le x < 1\}$. For a function on $\mathbb{T}$ the *n*th complex Fourier coefficient is defined by

$$\hat{f}_n = \int_0^1 f(x)\, e^{-2\pi inx} dx \tag{1.1}$$

and the formal series

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e^{2\pi inx} \tag{1.2}$$

is called the Fourier series of f.

The integrals

$$a_n = 2\int_0^1 f(x)\cos(2\pi nx)\,dx \quad \text{and} \quad b_n = 2\int_0^1 f(x)\sin(2\pi nx)\,dx \tag{1.3}$$

are called the *n*th cosine and sine Fourier coefficients, respectively. By the Euler formula $e^{ix} = \cos x + i\sin x$, (1.2) can be formally written as

$$f(x) \sim \frac{a_o}{2} + \sum_{-\infty}^{\infty}\left(a_n \cos 2\pi n x + b_n \sin 2\pi n x\right) \tag{1.4}$$

History of Fourier series: Fourier gave many examples of such representations in his book “Analytical Theory of Heat” (*Théorie Analytique de la Chaleur*, 1822), which contains also some of his previous works (1807, 1811), and made heuristic use of trigonometric expressions for a wide classes of functions (see for the history of Fourier series J.-P Kahane and P.-G. Lemarié-Rieusset[?])

Any periodic function with period 1 is identified with a function in the interval $\mathbb{T}$ and vice versa.

Change of scale: If $g(x)$ is a periodic function with period *T*, then the function $f(x) = g(Tx)$ is periodic with period 1. By definition

$$a_n = 2 \int_0^1 f(x)\cos(2\pi n x)\,dx = \frac{2}{T} \int_0^T g(t)\cos\frac{2\pi n t}{T}\,dt,$$

and

$$b_n = 2 \int_0^1 f(x)\sin(2\pi n x)\,dx = \frac{2}{T} \int_0^T g(t)\sin\frac{2\pi n t}{T}\,dt,$$

and the Fourier series of $g(t)$ is given by

$$g(t) \sim \frac{a_o}{2} + \sum_{-\infty}^{\infty}(a_n \cos\frac{2\pi n t}{T} + b_n \sin\frac{2\pi n t}{T}),$$

which is formally obtained from (1.4) by the change of variable $x = t/T$

Lebesgue spaces: For $1 \le p < \infty$ the Lebesgue space $L^p(\mathbb{T})$ is the collection of all measurable periodic functions f with period 1 such that $\|f\|_p = (\int_0^1 |f(x)|^p\,dx)^{1/p} < \infty$. For $f \in L^1(\mathbb{T})$ the Fourier coefficients are legitimately defined. We shall use a similar notation $L^p(\mathbb{R})$ for functions of a real variable $x \in \mathbb{R}$, $L^p(\mathbb{R}^d)$ for functions in the *d*-dimensional Euclidean space $\mathbb{R}^d$, $L^p(d\mu)$ for a general measure $d\mu$ instead of the Lebesgue measure dx and the same notation $\|f\|_p$ for all cases.

2.2. Convolution and Fourier Series

For $f, g \in L^1(\mathbb{T})$ the convolution $f * g(x)$ is defined by
$f * g(x) = \int_0^1 f(x-t)g(t)\,dt$.

If $f, g \in L^1(\mathbb{T})$, then we have

i) $f * g(x) = g * f(x)$ by a change of variable.
ii) $f * g(x) \in L^1(\mathbb{T})$ and $\| f * g \|_1 \le \| f \|_1 \| g \|_1$.
iii) $\left| \hat{f}_n \right| \ge \| f \|_1$. Furthermore, $\hat{f}_n \to 0$ as $n \to \infty$ (the Riemann-Lebesgue theorem).
iv) $\widehat{(f * g)}_n = \hat{f}_n \hat{g}_n$. Thus $f * g(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}_n \hat{g}_n e^{2\pi i n x}$.

2.3. Pointwise Convergence of Fourier Series

Partial sums of Fourier series. The *n*-th partial sum of the Fourier series of f is given by

$$s_n(f(x)) = \sum_{m=-n}^{n} \hat{f}_m e^{2\pi i m x} = \int_0^1 f(x-y)\, D_n(y)\,dy \tag{1.5}$$

where $D_n(x) = \sum_{m=-n}^{n} e^{2\pi i m x} = \sin \pi(2n+1)x / \sin \pi x$. The function $D_n(x)$ is called the *n*th Dirichlet kernel. The second equality follows from the fact that $\int_0^1 f(x-y) D_n(y)\,dy = \int_0^1 f(y)\; D_n(x-y)\,dy$ and that the kernel $D_n(x-y)$ is a sum of $e^{2\pi i m x} e^{-2\pi i m y}, m = 0 \pm 1, \pm 2, \ldots, \pm n$. By the periodicity of the integral domain [0,1) can be replaced by [-1/2,1/2).

Integral representation of partial sums: Put $\psi_x(y) = [f(x+y) + f(x-y) - 2f(x)]/2$. Then from (1.5) the following formula is obtained

$$s_n(f(x)) - f(x) = \int_0^{1/2} \psi_x(y) \frac{2 \sin \pi (2n+1) y}{\sin \pi y}\,dy$$

Localization theorem of Riemann: Let $f \in L^1(\mathbb{T})$ and fix any point x. A necessary and sufficient condition that the partial sum $s_n(f(x))$ converges to $f(x)$ is that

$$\int_0^{\delta} \psi_x(y) \frac{\sin \pi (2n+1) y}{\sin \pi y}\,dy \to 0 \quad \text{as } n \to \infty \tag{1.6}$$

for some $\delta > 0$. Thus the convergence at a point depends only on the property of f in a neighborhood of the point.

Test for convergence of Fourier series: The condition (1.6) is satisfied if one of the following holds:

(i) (The Dini-Lipschitz condition). Suppose that there exists $\delta > 0$ such that $\int_0^{\delta} \left(|\psi_x(y)| / y \right) dy < \infty$ (the Dini condition), then the partial sum $s_n\left(f\left(x\right)\right)$ converges to $f(x)$. In particular, if f satisfies the Lipschitz condition of order $0 < \varepsilon \le 1$, that is $|f(x) - f(y)| < c|x-y|^{\varepsilon}$ with a constant $c > 0$ then $s_n(f)$ converges uniformly to f.

(ii) (The Dirichlet-Jordan condition). If the function $f(x)$ is of bounded variation in the neighborhood of x and $\psi_x(y) \to 0$ as $y \to 0$, then $s_n(f(x))$ converges to $f(x)$.

(iii) (Bernstein's theorem): If f satisfies the Lipschitz condition of order $1/2 < \varepsilon \le 1$, then $\sum_{n=-\infty}^{\infty} \left| \hat{f}_n \right| < \infty$ and thus the Fourier series is absolutely convergent

Gibbs's phenomenon: Suppose that a function f is of bounded variation in the neighborhood of a point x and continuous except at x. Let $2d = f\left(x+0\right) - f\left(x-0\right) > 0$ and $f\left(x\right) = [f\left(x+0\right) + f\left(x-0\right)]/2$. Put

$$G = \frac{2}{\pi} \int_0^{\pi} \frac{\sin x}{x} dx = 1.17897975...$$

Then for any a satisfying $f(x) - Gd \le a \le f(x) + Gd$, there exists a sequence x_n tending to x such that $\lim_{n \to \infty} s_n(f)(x_n) = a$.

Convergence almost everywhere. The following are some remarkable results on the convergence (divergence) of Fourier series for the Lebesgue measurable functions.

(i) There exists a continuous function f such that the partial sum $s_n(f(x))$ diverges at $x = 0$ as $n \to \infty$ (du Bois Reymond, 1876). This result is ultimately strengthened by J.-P. Kahane and Y. Katznelson (1965): For any set *N* of measure 0 there exists a continuous function whose partial sum diverges on *N*.

(ii) There exists an integrable function f such that the Fourier series diverges everywhere (A.Kolmogorov, 1923).

(iii) If $f \in L^2(\mathbb{T})$, then $s_n(f(x))$ converges a. e. to $f(x)$, that is, converges everywhere except for a set of measure zero. (L. Carleson[?], 1966).
More precisely, Carleson's theorem holds for functions f such that $\int_0^1 |f(x)| \log^+ |f(x)| \log^+ \log^+ 1\infty$, where $x^+ = \max(x, 0)$ (N. Yu. Antonov,1996).

Summability kernel: A family of functions $\{k_\lambda(x) : \lambda > 0\}$ in $L^1(\mathbb{T})$ is called a summability kernel if it satisfies the conditions: (i) $\int_{\mathbb{T}} k_\lambda(x) dx = 1$, (ii) $k_\lambda(x)| \le c\lambda$ for

$|x| \le 1/2$, and (iii) there exists a constant $\eta > 0$ such that $|k_\lambda(x)| \le c/\lambda^\eta |x|^{1+\eta}$), where c is a constant independent of λ and x.

Summability: Under these three conditions, if $f \in L^1(\mathbb{T})$ and if x is a Lebesgue point of f that is, $h^{-1}\int_{-h}^{h}|f(x) - f(x-t)|\,dt \to 0 (h \to 0)$, then $k_\lambda * f(x)$ converges to $f(x)$ as $\lambda \to \infty$

It is known that is a function is Lebesgue integrable, then almost every point is a Lebesgue point.

If f is a continuous periodic function then $k_\lambda * f$ converges uniformly to f.

Approximate identity: The family $\{k_\lambda\}$ is called an approximate identity in $L^p(\mathbb{T})(1 \le p \le \infty)$ if $\|f - f * k_\lambda\|_p \to 0$ as $\lambda \to \infty$ for every $f \in L^p(\mathbb{T})$. A summability kernel is an approximate identity in $L^p(\mathbb{T})(1 \le p < \infty)$ and in the space $C(\mathbb{T})$ of continuous functions.

Examples of summability kernels: The Fejér kernel which is defined by

$$F_n(x) = \sum_{m=-n}^{n}(1 - \frac{|m|}{n+1})\, e^{2\pi imx} = \frac{1}{n+1}(\frac{\sin \pi(n+1)x}{\sin \pi x})^2,$$

is a summability kernel with $\eta = 1$ and $\lambda = n$. The Poisson kernel $P_r(x)$ given below is another example of summability kernel with $\eta = 1$ and $\lambda = 1/(1-r)$.

2.4. Norm Convergence of Fourier Series.

Parseval's formula: If $f \in L^2(\mathbb{T})$, then

$$\|f\|_2 = (\sum_{n=-\infty}^{\infty}\left|\hat{f}_n\right|^2)^{1/2}$$

Conversely if $\{c_n \in l^2(\mathbb{Z})$, that is, $\sum_{n=-\infty}^{n}|c_n| < \infty$, then there exists a function f in $L^2(\mathbb{T})$ such that $f(x) \sim \sum c_n e^{2\pi inx}$ (Riesz-Fischer theorem}. This implies that the mapping of $f \in L^2(\mathbb{T})$, then the partial sum $s_n(f)$ converges to f in L^2

Conjugate function: For f in L^1 the series

$$\sum_{n=-\infty}^{\infty} \left(-i \operatorname{sign} n\right) \hat{f}_n e^{2\pi int}$$

is called the conjugate Fourier series of f where sign $z = 0$ $(z = 0)$ and $= z/|z| (z \neq 0)$. If $f \in L^1(\mathbb{T})$, then it converges in a certain sense to a function $\tilde{f}$ called the conjugate function (see Section 1.5).

M Riesz theorem:

i) If $f \in L^p(\mathbb{T}), 1 < p,$, then $\|\tilde{f}\|_p \leq c_p \|f\|_p$ (M.Riesz), and

ii) If $f \in L^1(\mathbb{T}), \sup_\lambda \lambda m\left(\left\{x \in \mathbb{T} : \left|\tilde{f}(x)\right| > \lambda\right\}\right) \leq c_1 \|f\|_1$ (A.Kolmogorov)

where the best possible constant c_p is given by $c_p = \tan(\pi/2p)$ for $1 < p < 2, = \cot(\pi/2p)$ for $2 < p < \infty$ (S.Pichorides), and = $(1 + 1/3^2 + 1/5^2 + \ldots) / (1 - 1/3^2 + 1/5^2 - \ldots)$ for $p = 1$ (B.Davis).

Norm convergence: M. Riesz's theorem implies that if $f \in L^p(\mathbb{T}), p > 1$, then $s_n(f)$ converges to f in L^p, that is, $\|s_n(f) - f\|_p \to 0$ as $n \to 0$. If $f \in L^1(\mathbb{T})$, then $s_n(f)$ converges to f in measure, that is, for every $\lambda > 0, m\left(\left\{x \in \mathbb{T} : \left|s_n(f(x)) - f(x)\right| > \lambda\right\}\right) \to 0$ as $n \to \infty$ by Kolmogorov's inequality (ii)

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Bibliography

[Ca1] L.Carleson, Convergence and growth of partial sums of Fourier series, Acta math. **116** (1996), 135-157. [This is the first paper that solved the a. e. convergence problem of Fourier series of functions in L^2.]

[Da1] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992. [This book presents a wide view of mathematical theory of wavelets and its applications.]

[De1] L. Debnath, Integral Transforms and their applications, CRC Press, 1995. [This text book presents all essential aspects of integral transforms.]

[ER1] D.F. Elliott and K.R.Rao, Fast Transforms Algorithros, Analysis, Applications, Acad. Press 1982. [This book presents several kinds of the Fast Fourier transforms and other fast transforms.]

[Er1] A. Erdeli, Higher Transcendental Functions, Vols. 1 and 2, McGraw-Hill, New York 1953. [This and the next books are big collection of mathematical formulas and widely used.]

[EMOT1] A. Erdeli, W. Magnus, F. Oberhettinger, and F. Tricomi, Tables of Integral Transforms, Vols. 1 and 2, McGraw-Hill, New York 1954.

[GR1] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 1980. [This book is a handy and useful collection of mathematical formulas.]

[HW1] E. Hernández and G. Weiss, A First Course on Wavelets, CCR Press, 1996. [A plain introduction to the wavelet theory and function spaces]

[KL1] J.-P. Kahane and P.-G. Lemarie-Rieusset, Fourier Series and Wavelets, Gordon and Breach PubI. 1995. [The first part presents a Fourier series theory with historical description and the second parts wavelet theory.]

[MC1] Y. Meyer, Ondeletis et Operateurs, Vols. I, II, II(with R. Coiffman), Hermann, Paris, 1990. English transI. I; Wavelets and Operators, Cambridge University Press, 1993. II, III; Wavelets: Calderon-Zyground and Multilinear Operators, ibid., 1977. [This book presents the general theory of wavelets as well as applications to the theory of singular integrals, and function spaces.]

[St1] E.M. Stein, Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press, 1993. [This book includes the recent development of Fourier analysis, real variable theory and their applications.

[Zy1] A. Zygmund, Trigonometric Series. Vols. I and II, Cambridge Univ. Press, 1959. [A monumental book of the theory of Fourier series]