CONSTRUCTION OF RANDOM FUNCTIONS AND PATH PROPERTIES

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Contents

Summary

A few examples (Poisson process, Markov chain, martingale, Brownian motion) of stochastic processes are given. On the general theory of stochastic processes Kolmogorov's existence theorem is formulated. Somewhat more detailed studies of the Poisson process and Brownian motion are presented.

1. Examples

1.5. Poisson Process

Consider a certain mass of radioactive substance and assume

(a) If $0 < t_1 < t_2 \le t_3 < t_4$ and $A_k(s,t)$ denotes the event that during the time interval (s,t) k disintegrations occur then the events $A_k(t_1,t_2)$ and $A_\ell(t_3,t_4)$ are independent for all nonnegative integer values of k and ℓ.

(b) The events $A_k(s,t)$ $k = 0,1,2,\dots$ form a complete system i.e.
$\bigcup_{k=0}^{\infty} A_k(s,t) = \Omega$ (Ω is the "sure event").

(c) If k is given $\mathbf{P}\{A_k(s,t)\}$ depends only on the difference $t–s$. Let $\mathbf{P}\{A_k(s,t)\} = P_k(t–s)$.

(d) If t is small enough, the probability, that during a time interval of length t there occur disintegrations > 1, is negligibly small compared to the probability that occurs exactly one. That is

$$\lim_{t\to 0}\frac{1-P_0(t)-P_1(t)}{P_1(t)}=0.$$

Let $\pi(t)$ be the number of disintegrations during the time interval $(0, t)$. Then clearly

$$\mathbf{P}\{\pi(t)=k\}=P_k(t).$$

$\pi(t)$ ($t \ge 0$) is an integer valued random process. It is a special stochastic process called Poisson process.

We obtain the same mathematical model investigating a number of practical situations. For example let $\pi(s,t)$ be the number of arrivals at a bank in (s,t). Then condition (a) means that the numbers of arrivals in disjoint intervals are independent, (b) means that the number of arrivals in a finite interval is finite. Condition (c) holds if the process is homogeneous in time. Finally (d) holds if two different customers cannot arrive in the very same moment.

As a further example consider the number of red blood cells in a microscope. It is natural to assume that the numbers of cells in disjoint domains are independent. The probability that a domain contains k cells depends only on the area of the domain but not on its location and a small domain does not contain more than one cell. Hence we have the above conditions on a plane, instead of the time axis. However, the fundamental properties of this process are the same in any dimension.

1.6. Markov Chain

Let $X_1, X_2, \dots$ be a sequence of random variables taking the values $1,2,\dots$ i.e. we assume that

$$\sum_{\mathrm{j}=1}^{\infty}\mathbf{P}\{X_{\mathrm{i}}=j\}=1 \quad (i=1,2,\dots).$$

Such a system is also a stochastic process. It is called a Markov chain if

$$\mathbf{P}\left\{X_n=j_n \middle| X_{n-1}=j_{n-1},\dots,X_1=j_1\right\}=\mathbf{P}\left\{X_n=j_n \middle| X_{n-1}=j_{n-1}\right\} \quad (1)$$

where $n = 1,2,\ldots$ and $j_1, j_2, \ldots j_n$, is an arbitrary sequence of integers.

Markov chains are usually interpreted as follows:

Let S be a physical system which can be in the states $A_1, A_2, \ldots$. Let the state of the system change at time $t = 2,3,\ldots$ and put $X_i = j$ if at time i the system is in state j.

The hypothesis (1) that the random changes of state of a system S form a Markov chain can be expressed as follows: The past of the system can influence its future only through its present state.
The Markov chain is a natural model of the "random walk". Consider a tourist standing at a corner of a town. He chooses randomly one of the possible directions (streets) and goes one block in the chosen direction. Arriving at the next corner he chooses again randomly the next direction without remembering his previous walk. The corners are numbered as 1,2,… and $X_1, X_2, \ldots$ is the sequence of the corners visited by the tourist. Condition (1) expresses that the tourist cannot remember his previous tour.

1.7. Martingale

Let $X_1, X_2, \ldots$ be a sequence of random variable staking the values $0, \pm 1, \pm 2, \ldots$ i.e. we assume that

$$\sum_{j=-\infty}^{+\infty} \mathbf{P}\{X_i = j\} = 1 \qquad (i = 1, 2, \ldots).$$

The sequence $\{X_n\}$ can be thought of as the fortune at time n of a player who is betting on a fair game. The fairness of the game is expressed by the following condition

$$\mathbf{E}\left(X_{n+1} \middle| X_1, \ldots, X_n \right) = X_n \tag{2}$$

i.e. the expected value of the fortune of the player after the (n+1)-th game is equal to his fortune after the n-th game.

The sequence $\{X_n\}$ satisfying (2) is called a martingale.

1.8. Brownian Motion

The English botanist Robert Brown observed in 1828 that microscopic particles suspended in a liquid are subject to continual molecular impacts and execute zigzag movements. For sake of simplicity now we consider the case of the linear motion. Let $\{W(t), t \geq 0\}$ be the location of the particle at time t and assume that

$$W(0) = 0,$$

$$\mathbf{P}\{W(t) - W(s) < x(t-s)^{1/2}\} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-u^2/2} du \quad (0 \leq s < t < \infty)$$

i.e. $W(t) - W(s)$ is normally distributed,

$W(t_2) - W(t_1)$, $W(t_4) - W(t_3), \ldots, W(t_{2n}) - W(t_{2n-1})$ are independent random variables whenever

$$0 \le t_1 < t_2 \le t_3 < \ldots \le t_{2n-1} < t_{2n} (n = 2, 3, \ldots),$$

$W(t)$ is a continuous function with probability 1.

The stochastic process $\{W(t), t \geq 0\}$ is called Brownian motion or Wiener process. We note that the Brownian motion is the most studied and applied continuous stochastic process. For example in Bachelier's work (1900) it is a model of stock market.

2. Definition of the Stochastic Process

A stochastic (or random) process $X(t)$ is formally defined to be a collection of random variables and indexed by the elements of a parameter set $t \in T$. The set T generally is one of these

$$\mathbb{R}^1 = (-\infty, +\infty), \qquad \mathbb{R}^+ = (0, \infty),$$

$$\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}, \qquad \mathbb{Z}^+ = \{0, 1, 2, \ldots\}, \qquad \mathbb{R}^d,$$

where $\mathbb{R}^d$ is the d-dimensional Euclidean space. In Examples 1.1, 1.2, 1.3, 1.4 the parameter sets T are $\mathbb{R}^+$, $\mathbb{Z}^+$, $\mathbb{Z}^+$, $\mathbb{R}^+$ respectively. The values of the random variables generally are taken from $\mathbb{R}^1$. It is the case in Example 1.4, in Examples 1.1, 1.2 the set of possible values is $\mathbb{Z}^+$, in Example 1.3 it is $\mathbb{Z}$. In a number of applications it is worth-while to consider more general sets of possible values e.g. $\mathbb{R}^d$. The set of possible values is also called state-space.

The stochastic process $X(t)$ can be considered as a random function defined on T. However, one has to remember that $X(t)$ is a random variable for each $t \in T$. Whenever we consider $X(t)$ as a random function i.e. as a function obtained by a random experiment then $X(t)$ is called a trajectory or a path-function or a sample-function of the process.

In most cases the stochastic processes are described by their so-called finite-dimensional distributions. Let $t_1, t_2, \ldots, t_n \in T$, $n = 1, 2, \ldots$ and $x_i \in \mathbb{R}^1$ $(i = 1, 2, \ldots, n)$. Then the distribution functions

$$\mathbf{P}\{X(t_1) < x_1, X(t_2) < x_2, \ldots, X(t_n) < x_n\} = F_{t_1, t_2, \ldots t_n}(x_1, x_2, \ldots, x_n)$$

are the finite-dimensional distributions of the process $X(t)$. (This definition can be applied only when the state-space of $X(t)$ is $\mathbb{R}^1$, or a subset of that. In case of more general state spaces similar definition can be given.)

Clearly the distribution functions $F_{t_1,t_2,\ldots t_n}(x_1, x_2, \ldots, x_n)$ should satisfy the following conditions

$$F_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_{i-1}, \infty, x_{i+1}, \ldots, x_n)$$
$$= F_{t_1,\ldots,t_{i-1},t_{i+1},\ldots t_n}(x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_n), F_{\pi(t_1,\ldots t_n)}(\pi(x_1, \ldots x_n))$$
$$= F_{t_1,\ldots,t_n}(x_1, \ldots x_n)$$

where π is an arbitrary permutation of 1,2,…,n.

A very fundamental theorem of the theory of stochastic processes is due Kolmogorov telling us:

If $F_{t_1,t_2,\ldots t_n}(x_1, x_2, \ldots, x_n)$ *is a family of distribution functions satisfying the above two conditions then there exists a stochastic process* $X(t)$ *having these finite dimensional distributions.*

Note that Kolmogorov's theorem claims only existence, not uniqueness, which is even not true in general.

Kolmogorov's theorem easily implies the existences of the processes described in Examples 1.1, 1.2, 1.3. The proof of the existence of the Brownian motion is much harder. In fact Kolmogorov's theorem implies the existence of a process satisfying the first three conditions of the Brownian motion from section 1.4. However, to prove the continuity is much harder.

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Biographical Sketch

P. Révész was born on June 6, 1934 in Budapest and received the Ph.D. degree in 1958. Dr. Révész was Associate Professor at the University of Budapest during 1957-1964, Fellow, Mathematical Institute of the Hungarian Academy of Sciences, 1964-1984, Full Professor, Vienna University of Technology, 1985-1980, and retired in 1998. Dr. Révész is a Member of Hungarian Academy of Sciences, Academia Europaea, International Statistical Institute, Institute of Mathematical Statistics, and Bernoulli Society (President: 1983-85). Dr. Révész's publications include *The laws of large numbers* (Academic Press, New York 1967), *Strong approximations in probability and statistics* (Academic Press, New York 1981 jointly with M. Csörgö), *Random walk in random and non-random environments* (World Scientific, Singapore 1990), and *Random walks of infinitely many particles* (World Scientific, Singapore 1994)