MODELS AND METHODS OF ACTUARIAL MATHEMATICS

V.I.Arkin
Central Economic Mathematical Institute of Russian Academy of Sciences, Moscow, Russia

S.Ya.Shorgin
Institute of Informatics Problems of Russian Academy of Sciences, Moscow, Russia

Keywords: Insurance, actuarial mathematics, risk, claim, individual risk model, collective risk model, premium, “ruin” probability, estimations, factorization model.

Contents

Summary

Insurance is a social mechanism that allows individuals and organizations to compensate economic losses caused by unfavorable events. Actuarial mathematics is the mathematical theory of insurance. There exist numerous mathematical models of insurance company activity; main models are collective risk model and individual risk model. The traditional tasks of actuarial mathematics are evaluation of insurance

premiums, "ruin" probability, distribution of surplus and total amount of claims.

1. Introduction

1.1. Brief historical overview and basic concepts of insurance and actuarial mathematics

The institution of insurance is as old as the civilization itself. Initially, it took the form of mutual assistance; contracts for mutual economic assistance were common in the field of wandering trade. As the techniques of insurance were perfected, the system of regular premium payments appeared. It enabled the accumulation of money into an insurance fund. Hence emerged the problem of determination of premium payments acceptable for all participants of the insurance process – a typical problem of actuarial mathematics. Such problems were already present in Ancient Roman practice of mutual insurance in professional corporations. Similar schemes and related problems were common to Medieval guilds.

Transition to market economy implies commercialization of insurance, as it undergoes a transformation from a "fraternal" mutual assistance into an economic good that is bought and sold; the transformation is followed by the appearance of two asymmetric parties – the insurer who offers insurance coverage for sale, and the insurer, who buys that good. Under market conditions, optimization of insurance premiums, now playing the role of the price of the insurance package, became even more pertinent problem; these premiums had to satisfy the conditions of moderateness and sufficiency (see below).

The determination of insurance tariffs or premiums became one of the key problems facing the emerging science of actuarial or insurance mathematics. In its initial stages, the key problem was the determination of life insurance tariffs.

In Ancient Rome, the word *actuaries* denoted, on one hand, persons responsible for the writing-down of Senate acts, on the other hand – officers who were responsible for accounting and supplies in the army. In the English language the word *actuary* changed its meaning several times. First, it denoted registry clerks, later – secretaries and advisors in corporations, especially in insurance firms. With time, the word *actuary* came to denote those who carried out mathematical computations connected with the determination of the probability of certain life expectancy – the basis for life insurance contracts.

For a long time actuarial computations and methods of actuarial mathematics were associated solely with the field of life insurance. However, the development of insurance business caused by the industrial revolution and trade boom in the 18-20th centuries demanded the development of other methods of insurance tariff computation applicable to the field of non-life insurance. At the beginning of the 20th century the accepted opinion was that the progress of insurance business implies the replacement of primitive constructs with precise, scientifically grounded "system of probability" in the process of insurance tariff determination.

In the modern understanding, actuarial mathematics or mathematical theory of risk can be seen as the system of mathematical methods and results that allow a qualitative description of the relationship between insurance organizations and their clients (the insured) (the word *actuary* now denotes an expert in "insurance mathematics"). In the nexus of all actuary problems lies the presence of uncertainty. Although actuary mathematics was created as an amalgamation of several branches of mathematics, the principle rope in that science is played by the methods and results of probability theory.

A.N.Shiryaev divides the history of actuarial mathematics into three stages:

- the first stage can be traced to Galley, who in 1693 composed tables for life insurance. His opinion was that the number of deaths in each group is predetermined;
- the second stage is tied to the introduction of probabilistic ideology and statistical methods into life and other forms of insurance. This stage is by no means over;
- the third stage, which began recently, is characterized by widespread use of financial instruments and financial engineering for risk minimization.

Below the main stage of actuarial mathematics, i.e. the second stage, is considered.

1.2. Insurance risks

Insurance is a social mechanism that allows individuals and organizations to compensate economic losses caused by unfavorable events. According to A.N.Shiryaev, insurance is called upon to replace uncertainty about possible future losses with certainty. The most efficient method of reducing losses from uncertainty is the employment of a collective mechanism that would distribute the losses of an individual among all the participating members.

It would be inappropriate to attempt to reduce insurance (and the corresponding mathematics) to only to life insurance or to some other kind of insurance. Insurance should be understood in a broader sense – as risk insurance. Although life insurance has its specific character rooted in the historical traditions of the field and tied with certain demographic aspects (mortality tables, etc.), the general problems of insurance and the corresponding mathematical tools remain the same for all types of insurance. Because of this, and also due to the volume limitations, specifics of life insurance and the corresponding actuary methods are outside the scope of this article. Briefly, it can be said that the specifics of life insurance are tied with the demographic nature of the distribution of claims and with accumulation (interest rate).

The source of risk and accompanying losses are random events that fall into two broad categories – physical and moral uncertainty.

Moral uncertainty implies random events related to certain human behavioral traits – dishonesty, carelessness and such. The corresponding risks are not insurable.

Although insurance is an effective risk aversion mechanism, it cannot compensate for all uncertainties and related losses. In order to be insurable, risk must satisfy several conditions, namely:

- there should exist a large uniform group of the insured, whose traits are to some degree preserved in time;
- possible losses should not simultaneously affect a large number of the insured;
- the causes and magnitude of losses should not be determined by premeditated actions of the insured; they should be random;
- the losses should be easily identifiable and difficult to simulate;
- potential losses should be significant to make insurance reasonable;
- the probability of the occurrence losses should be small to make insurance affordable;
- there should exist a body of real-life data which can serve as a basis for the computation of risk.

1.3. Insurance as a risk diversification mechanism

From the viewpoint of modern economic science, insurance is often seen as an economic mechanism for risk diversification.

The first type of such mechanisms includes variations of reserve mechanisms, when an economic agent creates a reserve (in goods or currency) that can be used to offset unexpected economic damage or variations in income. The optimal saving policy is reduced to finding the optimal spending to reserve ratio at every moment in time.

It is significant that this mechanism can be used by an independent economic agent. The behavior of that agent has been described by multiple economic models that determined optimal saving as well as optimal consumption strategies. The same rules govern the mechanism of lending to an economic agent. All these mechanisms imply time diversification of risk (or risk premium).

At the same time, the behavior of banks that simultaneously lend to many economic agents is governed by a different type of mechanism. It is a more complicated one, and includes various generalized insurance strategies that diversify risk among many economic agents participating in the process. The creation of a reserve occurs here as well, under different principles.

This kind of risk diversification is only possible if the number of risk-bearing economic agents is large. Risk diversification is carried out by the redistribution of economic damage caused by random deviations of individual economic factors from their mean. Usually this mechanism implies the creation of a stabilization fund maintained by insurance premiums by the participants.

It is significant that for most agents, the premiums of all sorts are lost forever – they serve only as a payment for certainty that any possible random damage will be compensated for (the situation with life insurance policy is different and is not covered in this article). There is no real stabilization of aggregate income – only the redistribution of damage taken by the “losers” among all participants, which in the end leads to the stable existence of the system as a whole.

Risk diversification mechanisms by themselves do not and cannot reduce the actual risk of economic damage but only redistribute the summarized risk. The effectiveness of such mechanisms is an increasing function of the number of participating economic agents and a decreasing function of the probability occurrence of individual economic damage.

Insurance contracting implies the existence of *a priori* information on the distribution of the magnitude of economic damage and the possibility of refining this information by observing the behavior of economic agents.

The role of insurance risk diversification mechanisms is usually served by:

- The creation of an insurance organization that promises to provide full or partial compensation from the pool of accumulated insurance premiums.
- The creation of a mutual insurance organization (in this case, compensation takes place through redistribution of the insurance fund).
- Reinsurance, that is, the redistribution of risk among insurance organizations through, for instance, resale of obligations to cover risks or through other contracts among insurance firms.
- Long-term contracts on the right to buy or sell (options).

The main topic of the article is the first of these mechanisms, which appeared first historically and has seen more research and received more attention from actuarial mathematical literature. In such a setup, the insurer sells the insurance policies for prices (premiums) that satisfy the following natural conditions: the premium should not be too large, so that the buyer is not discouraged (which is especially important if the competition is present), nor too small, so that the insurance fund is sufficiently large to make insurance payments. These conditions can be formalized in several ways; some of them are examined below.

The research of stochastic processes that take place in the insurance portfolio of an insurance organization (the set of insurance contracts made by that organization) under the assumption that both the number and magnitude of insurance claims are random variables is in the domain of a division of actuarial mathematics – the mathematical theory of insurance risk.

The main practical task of risk theory is the estimation of probability of insurer's "ruin", when the initial capital invested in the insurance fund and the accumulated insurance premiums are not sufficient to cover all the insurance claims, and the definition of insurance tariffs acceptable for the insurer under a given probability of "ruin".

2. Empirical principles of determination of insurance premiums

Before one can discuss the problems related to modeling the risk process of an insurance company, some attention must be paid to "empirical" principles of determination of insurance premiums from probabilistic characteristics of total claims (losses), often used in actuary practices. Usually, these principles are based on some "common sense" logic rather than upon detailed modeling of processes of premium cash inflows and claim cash outflows. The time distribution of total losses (which is random, of course) is assumed to be known *ex ante*.

Following Rotar, here are several widely used empirical principles.

Suppose that D is the value of the premium, X is the random value of the claim with the distribution function $F(x)$. D is a real-valued functional on the set of distribution functions dependent on some exogenous variable λ that determines the rule of the choice of premium.

Thus,

$$D = \pi(F, \lambda),. \tag{1}$$

where $\pi(F, \lambda)$ is some function. Let us consider several well-known principles for choosing $\pi(F, \lambda)$.

1. Expected value principle:

$$D = (1+\lambda)\,\mathbf{E}X,\ \lambda \geq 0. \tag{2}$$

($\mathbf{E}$ denotes the expected value of a random variable).

In this case, λ is called the load coefficient since it denotes the difference between the insurance premium and the average claim. When $\lambda = 0$ one can speak of the net premium principle when the average losses of the insurer equal to the insurance premium. This principle is often used in life insurance.

2. Variance principle:

$$D = \mathbf{E}X + \lambda\,\mathbf{D}X,\ \lambda > 0. \tag{3}$$

($\mathbf{D}$ denotes the dispersion of the corresponding random variable).

Here, λ plays the role of weight coefficient for the dispersion – the higher is λ, the stronger is the relation between premium and dispersion of claims.

3. Standard deviation principle:

$D = \mathbf{E}X + \lambda\sqrt{\mathrm{DX}}$, $\lambda > 0$. Here, the meaning of λ is the same as above. This principle, as will be seen below (Sections 5.4 and 5.6), can be grounded in the framework of individual risk model under the assumption of normality of total losses.

4. Zero utility principle:

This principle is based on the expected utility theory developed by J. von Neumann and O. Morgenshtern. The key role in this theory is played by the utility function – a concept that can be traced to D. Bernoulli. It is assumed that "utility" or the "satisfaction experienced by an individual (or a group of individuals) from a certain income of y increases not proportionally to y, but according to some nonlinear function $u(y)$. Thus, the experience of an extra dollar is different for an individual with a capital of a million dollars and for one with a capital of one dollar. One can assume, for instance, that marginal utility is proportional to relative changes in income, that is $du = k\, dy / y$. That leads to $u(y) = k \ln(y)$ + const.

Suppose that $u(y)$ is the insurer's utility function which is assumed to be strictly increasing and convex. If $S \geq 0$ is the insurer's starting capital, premium D is the solution to

$$\mathbf{E}\, u(S + D - X) = u(S), \quad (4)$$

that is, the premium is chosen to equal the expected utilities before and after the insuring takes place. If the utility function is exponential,

$$u(y) = \mathrm{a}^{-1}(1-\mathrm{e}^{-\mathrm{a}y}), \mathrm{a} > 0, \quad (5)$$

the latter equation has a solution

$$D = \mathrm{a}^{-1} \ln(\mathrm{E} \exp(\mathrm{aX})). \quad (6)$$

This case is known as the exponential principle.

5. Generalized principle of zero utility:

Suppose that the starting capital S is a random variable. The insurance premium D is determined as a solution to

$$\mathbf{E}\, u(S+D-X) = \mathbf{E}\, u(S). \quad (7)$$

In this more general case D depends on the joint distributions of X and S.

6. Escher principle:

$$D = \mathbf{E}\,(X \exp(\lambda X))\,(\mathbf{E} \exp(\lambda X))^{-1},\ \lambda \geq 0. \quad (8)$$

The Escher principle arises during the from the minimization of firm’s expected losses if the loss function has the following form

$$L(x,D) = (D - x)^2 \exp(\lambda \mathrm{x}). \quad (9)$$

One can say that D is the average of X obtained by the multiplication of X by an increasing weight function, which makes a risky situation less attractive for the insurer. The Escher parameter λ reflects risk aversion of the insurer.

7. Swiss-principle:

$$\mathbf{E}\, f(X-\lambda D) = f((1-\lambda)D),\ \lambda \in [0,1], \quad (10)$$

where $f(x)$ is real-valued, continuous, strictly increasing and convex function. Note that for λ = 0 it transforms into the generalized average principle $D = f^{-1}(Ef(X))$, for λ = 1 – into the zero utility principle with the utility $u(z) = -f(-z)$. If $f(x) = a^{-1}(\exp(ax)-1)$, it transforms into the exponential principle, if λ = 1 and $f(x) = x\exp(ax)$ – into the Escher principle.

8. Orlicz principle:

In this case, *D* is the solution to

$\mathbf{E}\,\rho(XD^{-\lambda}) = \rho(D^{1-\lambda})$, where $\lambda \in [0,1]$ and ρ - is continuous and strictly increasing function. With λ = 0 one obtains the generalized average principle.

All these principles of premium determination are not related to the generation of the random variable of the insurer's losses, that is, they do not consider the processes of inflow of premiums and outflows of claims. In this sense these principles should be viewed as empirical. Some of them (like the standard deviation principle; see below) are theoretically grounded under certain assumptions about the models of insurer's risk processes. The generation of such models is one of the main problems facing modern actuarial mathematics. In the following sections several such models will be considered.

-
-
-

Bibliography

Borch, K. (1974). *The Mathematical Theory of Insurance*. – Lexington Books. D.C.Heath and Company. Lexington, Massachusetts. Toronto. London. 373 p. [One of the most interesting monographs in the field of modern actuarial mathematics on the topic of neoclassic insurance].

Bowers, N.L., H.U. Gerber, J.C. Hickman, D.A. Jones, C.J Nesbitt (1986). *Actuarial Mathematics*. – Published by The Society of Actuaries, Itasca, Illinois. 624 p. [One of the most popular and detailed monographs, covering almost the entire field of actuarial mathematics. The questions of life and "Non-Life" insurance are addressed, as well as the problems of individual and collective risk theory. The work serves as a standard preparation guide for actuarial exams].

Embrechts P., Kluppelberg C. (1993). *Some aspects of insurance mathematics* (in Russian), Teoriya veroyatnostej i ee primeneniya 38, N 2, pp.375-416. [The intention of this work is to demonstrate the application of probability theory and mathematical statistics to insurance-related problems. The profit-making aspects of insurance and statistical aspects of insurance mathematics are also considered].

Grandell, J. (1991). *Aspects of Risk Theory*. - N.Y.: Springer-Verlag. 175 p. [In the study the main focus is the following generalization of the classic risk model: the occurrence of the claims may be described by a more general point process than the Poisson process].

Kalashnikov V.V., Konstantinidis D. (1996). *Ruin probability* (in Russian), Fundamentalnaya i prikladnaya matematika 2, N 4 (1006), pp.1055-1100. [An overview of a large number of works on asymptotic and guaranteed estimates of "ruin" probability in Lundberg-Cramer type models].

Panjer, H.H and G.E. Willmot (1992). *Insurance Risk Models*. - Schaumburg, Illinois: The Society of Actuaries. 442 p [This book deals with statistical modeling of the sizes of insurance losses from real data. It includes a large amount of distribution theory in addition to statistical estimation procedures for data that is in the form often available from insurance companies. One of basic themes of the book is that the distribution of aggregate claims for an insurer can be computed by simple recursive algorithms for a very wide range of claim number models with any loss distribution].

Rotar, V.I. and V.E.Bening (1994). *An introduction to Mathematical Theory of insurance* (in Russian) *Obozrenie prikladnoj i promyshlennoj matematiki*. V.1. No.5, p.698-779. Publishing Company TVP, 832 p. [The intention of this work is the survey of well-known ideas and models of mathematical risk theory. The work gives a modern, broader approach to insurance – it is viewed as a stabilization mechanism that distributes risk among many participants of the economic process].

Biographical sketches

Arkin V.I. is Head of Department of Central Economic Mathematical Institute of Russian Academy of Science, Member of Econometric Society, and Member of Moscow Mathematical Society with professional interests in the theory of probability and mathematical economics.

Shorgin S.Ya. is Deputy Director of Institute of Informatics Problems of Russian Academy of Science, Prof. of Department of Plekhanov Russian Economic Academy, Member of Russian Actuarial Society, and Member of International Informatization Academy with professional interests in the theory of probability and development of mathematical models of complex systems including actuarial mathematics.