MATHEMATICAL MODELS IN ECONOMICS

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Contents

Summary

The chapter provides a concise discussion of the role of mathematics in economic theory. After a brief overview of the main applications of the mathematical methods to economic problems, the chapter focuses on general equilibrium theory in its atemporal and intertemporal versions and discusses the concepts of equilibrium and dynamics in natural sciences and economics. In this context, business cycles models of Keynesian inspiration and neoclassical models of equilibrium dynamics are outlined and compared. Special consideration is also given to overlapping-generations models and the questions of implicit dynamics and learning.

1. Introduction

A mathematical model of the economy is a formal description of certain relationships between quantities, such as prices, production, employment, saving, investment, etc., with the purpose to analyze their logical implications. Some of those relationships derive from empirical observation; others are deduced from theoretical axioms concerning the assumed behavior of a “rational” economic agent, the so-called *homo oeconomicus*. Assuming that no mathematical mistake is made, the relevance and importance of the conclusion of the analysis depend on the validity of the premises of the model and on our ability to find out all their consequences. No matter how sophisticated the mathematical methods employed in the analysis are, the value of its final results heavily depends on the basic hypotheses of the model. To put it bluntly, the colloquial maxim “garbage in, garbage out” holds in mathematical economics as well as in any other area of scientific investigation.

This obvious truth, however, is not an argument against the use of mathematics in economic theory. Intuition and common language are usually not sufficiently powerful tools to investigate difficult and complicated problems, where the consequences of assumption are far from evident and often counterintuitive. In those cases, verbal reasoning must be complemented and supported by mathematics. The economics Nobel laureate Paul Samuelson, once (1952) wrote that “you can become a great theorist without knowing mathematics…[but]…you will have to be much more clever and brilliant”. This point was argued very forcefully by a great contemporary mathematician, Jacob T. Schwartz, who wrote extensively on economic questions (see, (1961)). We shall paraphrase his main argument as presented in an unpublished paper, as follows: (i) because of mathematics’ precisely defined, formal character, *mathematical arguments remain sound even if they are long and complex*; (ii) *formalisms of mathematics discipline mathematical reasoning,* and thus stake out an area within which patterns of reasoning have a reproducible, objective character; (iii) the fact that in mathematics there exists a precise notion of formal proof, disagreement as to whether a certain proposition is true or false (from given axioms) cannot persist for long and thus *mathematicians can work as a unified international and intertemporal community*; (iv) common sense, which acts by an undisciplined principle of plausibility and intuitive appeal, cannot cope either with situations in which truth is even partly counterintuitive, or with situations in which ingrained familiarity has lent overwhelming plausibility to some established error.

Even though the use of mathematics in economic analysis always had in the past enthusiastic supporters as well as fierce opponents, there is today a broad consensus that the discipline imposed by mathematics on economic reasoning was a fundamental factor in the development of economic theory as a science. However, one often forgets that the power of “hard” sciences such as physics to generate objectively valid scientific propositions depends on the systematic use of two techniques, namely: (i) consistency, i.e., adherence to the set of formal rules provided by mathematics; and (ii) experimental verification through which *observable* propositions generated by a mathematically coherent theory are confronted with the real world. What makes economic theory still unsatisfactory as an empirically relevant science is not the fact that, in order to apply technique (i), much of the economic concepts and arguments are abstract and “cannot be translated into English” as Mashall once complained. It is rather the fact that the validity of so many of the propositions generated by the theory cannot or at any rate are not yet systematically subjected to the verdict of experimental verification.

The introduction of mathematical methods into economics has been a long process. Pieces of essentially mathematical reasoning applied to economic problems have been detected as far back in history as in Aristotle’s work (see Theocaris (1961)) and in the XVIII and early XIX centuries, outstanding mathematicians such as Bernoulli, Gauss, Laplace and Poisson developed truly mathematical models to discuss economic problems.

But the rise of mathematical economics in the modern sense is usually dated back to Cournot’s (1838) classical (and long neglected) research on microeconomic theory. The definite recognition, though by no means the uncontested acceptance, of the mathematical methods in economics is associated with the progressive dominance in the economic profession of the so-called neoclassical school and in particular with the theory of competitive general equilibrium, a development that was crowned by the publication of Leon Walras celebrated *Élements d’Économie Politique Pure* first published in 1874 (see, Walras (1926)). Since then mathematics has been applied to virtually all areas of economic research and the lag between the production of new results in mathematics (and statistics) and their application to economics have been declining steadily. Traditionally, most new ideas and results in mathematics were inspired either by previous mathematical results or by problems suggested by physics. But because problems posed by economic theory (and economic life) are mathematically far from trivial, it became more common for mathematicians (or mathematical economists) to develop new ideas and methods, or extend and modify the known ones, in order to tackle them. Outstanding examples of this phenomenon are provided by the works of John von Neumann on economic growth and game theory, Kenneth Arrow and Gerard Debreu on general economic theory, John Nash on game theory and Frank Ramsey on optimal theory of saving. It is also well known that in the long gestation of his masterpiece *Production of Commodities by Means of Commodities,* Piero Sraffa availed himself of the counsel of first-rate mathematicians such as Abram Besicovitch and the already mentioned Frank Ramsey, although Sraffa himself went to an incredible length to avoid any difficult formalism.

At a lower level of sophistication, mathematical and geometrical tools were used to formalize Keynesian macroeconomic theory that dominated the field after the Great Depression, and to some extent as a reaction to it, till the 1970s. This is true, for example, of the so-called “IS-LM model”, first developed by John Hicks and Alvin Hansen in the 1930s, which soon became the standard formulation of macroeconomic equilibrium.

Finally, we should mention the econometric approach to the study economic data, the origins of which may be traced back to the seminal ideas of Evgeny Slutsky, George Udny Yule and Ragnar Frisch between the two world wars and was later developed, and given the status of orthodoxy, by the works of the Cowles Commission in the 1940s and 1950s.

2. Mathematics, General Equilibrium and Dynamical System Theory

Within the severe space limits of the present chapter, we cannot provide a detailed discussion of the many interesting questions and difficulties arising from the use of mathematical models in the various fields of theoretical economics. In the following pages, therefore, we shall focus on a small number of critical issues concerning the relation between mathematical economics, and in particular the theory of general equilibrium, and dynamical system theory – a broad, diverse and rapidly growing area of mathematical research.

A mathematician or a physical scientist typically looks at a dynamical system as a set of rules – the “laws of motion” – governing the evolution of certain state variables forward in time, from arbitrarily given initial conditions to any arbitrarily distant future. There are several different ways of modeling the evolution of systems in time but here we shall concentrate on two classes of models most commonly used in economic theory, namely difference equations and ordinary differential equations. (Besides difference and differential equations, ordinary or partial, there exist several other types of mathematical models representing dynamical phenomena in natural and social sciences, such as lattice maps, cellular automata, dynamical games, and so forth. Examples of applications to economics of some of these methods can be found in two recent books on complexity in economic systems, namely: Albin (1998) and Colander (2000).) Moreover, we will almost exclusively deal with deterministic system, i.e., systems that do not include random variables. The reader wishing to study a different approach to economic dynamics would greatly benefit from reading the book edited by Marimon and Scott (1999), which almost exclusively deals with stochastic (equilibrium) models.

Difference equations postulate a functional link between the values of state variables at different, discrete instants of time. The canonical form of discrete-time dynamical systems is:

$$x_{t+1} = G(x_t) \tag{1}$$

where $G: U \to \mathbb{R}^m$ with U an appropriate subset of $\mathbb{R}^m$, and $t \in \mathbb{Z}$. In this context, the terms "map", or "mapping" are often used for the function *G*. A closed-form solution of (1) starting at an initial point x_0 is a function $x(t; x_0)$ satisfying (1) for all $t > 0$ (or for *t* in some subset of $\mathbb{Z}$). Closed-form solutions (i.e., solutions written as combinations of elementary functions) are available only in special cases, e.g., when *G* is linear. However, given an initial point x_0, iterations of *G* generate sequences $(x_1, x_2, \ldots)$, that describe the future evolution of the system. (If the function *G* is invertible, iterates of $F \equiv G^{-1}$ generate the "past history" of the system as well). Typically, the dynamical properties of *G* depend on certain parameters that can sometimes be given an interpretation in terms of underlying (economic) problem.

Alternatively, we can use differential equations postulating a functional relation between the state of a system at a given time and its rate of change (velocity) at the same time. In this case, the canonical form is:

$$\dot{x} = f(x) \tag{2}$$

where $\dot{x} \equiv dx/dt; t \in \mathbb{R}; f: U \to \mathbb{R}^m$. Under generally conditions, the ensemble of solutions of (2) can be represented as a one-parameter family of maps $\phi(t, x)$ that, for each pair $(t, x) \in (I, U)$, where $I \subset \mathbb{R}$ is an interval of the real line, represents the state of the system starting at *x* after a time *t*. Once again, in the nonlinear case, only seldom can we actually write ϕ in closed-form.

When the laws of motion are written down (either in discrete- or in continuous time form), the purpose of the investigation is to study the properties of the orbits generated by the system, starting from some initial conditions arbitrarily chosen within a "reasonable" subset of the state space. There is a dilemma here. Linear systems are easier to study but they are morphologically poor. Nonlinear systems are infinitely more interesting, but correspondingly more difficult. Because exact, closed-form solutions are usually not available, in the nonlinear case we must adopt a strategy that combines analytical, numerical and graphical methods. In the study of the orbits of dynamical systems, special attention is given to certain invariant sets, such as fixed points, periodic, quasi-periodic and chaotic sets. The properties of those sets can be investigated from a geometric-topological point of view – for example, studying their stability and indecomposability or estimating their fractal dimensions. Or, we can take a metric-ergodic point of view and try to determine the statistical properties of ensembles of orbits – for example, finding invariant probability distributions supported by invariant sets, estimating metric entropy, etc. Besides studying the orbit structure of a system for a given, fixed set of parameters, we are also interested in discussing *variations* taking place in that structure when one or more parameters is changed, defining certain typical local or global bifurcations and their consequences.

This approach, that for short we shall henceforth label "the scientist's approach", philosophically takes the view that in the natural world causality flows forward in time, i.e., the past determines the future but not the other way around. The discovery of chaos has not changed this point of view, although it has cast doubt on the Laplacian belief that, given the initial conditions of a deterministic system, we can predict its dynamics accurately far into the arbitrarily remote future. As everybody knows, dynamical system theory in this sense has made extraordinary progress in the last decades of the 20th century and its applications cover all fields of natural sciences, engineering, biology, medicine and many social sciences as well.

Generally speaking, economic agents' decisions depend on past events and their expectations about the future. Moreover, it takes time to obtain and process information, decisions are not formed instantaneously, production involves a more or less long gestation period, and so on and so forth. Finally, economic activity is influenced by exogenous factors, such as exogenous technical progress that are in their turn taking place over time. These factors can be represented as given functions of time, stochastic processes, or a combination of the two. For all these and other analogous reasons, economic theory has an intrinsic intertemporal character and the investigation of this fundamental aspect of real economies has generated a large number of models aimed to describe their dynamics. Nevertheless, economists' opinions on the relevance and fruitfulness of dynamical system theory when applied to economic problems have fluctuated in time and, not surprisingly, they differ strongly between different schools of thought and areas of research. Thus, when the question is posed whether that formidable body of theoretical results and empirical findings is relevant to economics, the jury is still out.

In this respect a basic distinction exists between disequilibrium and equilibrium economic models, and therefore we should pause a moment to briefly discuss a fundamental difference in the way economists and scientists use the notion of equilibrium. To a dynamical system theorist, an equilibrium of system (1) is a fixed point, i.e. a value $\bar{x}$ of the state variable such that $G(\bar{x}) = \bar{x}$. Analogously, an equilibrium of system (2) is a value $\bar{x}$ such that $f(\bar{x}) = 0$. Thus, an equilibrium is a special state of a dynamical system such that if the system starts there and is not disturbed, it will remain there forever. As we shall see below in greater detail, to an economist equilibrium is a situation in which all markets "clear", i.e., demands and supplies generated by rational, optimizing agents are equal in all markets, and agents' expectations are always verified. In a deterministic context, this means that agents have "perfect foresight". The resulting values of the state variables (typically prices and quantities of commodities) are not necessarily constant in time – in this case we speak of stationary equilibrium – but they can be sequences of values that may or may not converge to a stationary state. Thus, the economist's equilibrium may or may not coincide with the mathematician's. In what follows, we shall use the term "equilibrium" in the economists' sense, reserving the term "fixed point" for equilibrium point in a mathematical sense (of course a stationary equilibrium is a fixed point).

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Biographical Sketch

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