METHODS OF TRANSFORMATION GROUPS

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Contents

Summary

The chapter is dedicated to a large branch of application of continuous transformation (symmetries) groups to the problems of mathematical physics. The focus is in particular on the applications and development of modern theoretical group methods. In this connection, the main moments of continuous group theory are given containing: definition of local Lie groups, Lie equations, invariant varieties, invariant and differential equations included; continuation of point transformations, invariant and partly invariant solutions. Then the elements of the theory of tangential transformations of finite and infi-

nite orders are given. On their basis the mutually unambiguous correspondence between non-trivial variational symmetries and non-trivial conservation laws (Neother theorem in modern interpretation) is established. The extremely important question of construction of the basis of conservation laws is discussed on the basis of common theory of Lie-Bäcklund groups and on concrete examples. The following methods of soliton mathematics are also treated: Method of Inverse Scattering Problem, Hirota transformation, Bäcklund, Penleve Property, Lax Pairs. As examples the following non-linear differential equations are analyzed: Korteweg de Vries (KdV) equation and its modifications, sine-Gordon equation, Schrödinger equation. All these are typical equations admitting soliton solutions.

1. Continuous Transformation Groups

The group analysis of differential equations has become nowadays a powerful instrument of research in non-linear problems. Its application to the fundamental equations of mechanics and physics is especially successful, as the principals of invariance are laid already during the deduction of these equations. A Norwegian mathematician Sophus started the systematic research of continuous Lie (1842 – 1899) transformation group in the second half the 19-th century. These groups called Lie groups represent the unification of algebraic group and topological structures connected by the demand of continuation of group multiplication operation. Thus the theory of transformation groups is on the junction of algebra and analysis. At the same time the value of this theory exceeds the limits of its application in mathematics itself. In fact the continuous transformation groups are the real objects of the world around us (environment), whose presence may be judged according to their influence on our ideas of its physical structure. Those are for example the ideas of homogeneity and isotropy of space and time, of dynamic similarity of phenomena of Galilee and Lorentz invariance, etc.

One of the principal problems of group analysis of differential equations is the studying of the activity of the group admissible by the given equation (or group of equations) on the set of solutions of this equation. The activity of the admissible group contributes to the set of solutions an algebraic structure that can be used with different aims. Among these aims the description of the common structure of the family of all the solutions, the apportionment of the definite class of solutions, the finding of which is easier to some extend than the common solution, the production of solutions from already made, etc.

For differential equations appearing in some mathematical model of the class of physical phenomena the admissible group with the model isn't usually given. That is why the pure technical problem of finding the largest group admissible by the given system of the equations is extremely important.

Another interesting and practically important problem consists in the use of the techniques of group analysis for the group classification of differential equations. The solution of this problem isn't only of pure mathematical interest, but also has an applied value. The question is that the differential equations of mathematical physics often contain parameters or functions that are found by experiment and therefore aren't strictly fixed – they are the arbitrary element of equation. At the same time the equations of mathematical model should be simple enough to be analyzed and solved successfully.

The group approach allows accepting as the criterion of simplicity the demand for the arbitrary element to be of the sort that with it the differential equation modeling would admit a group with the definite properties or even the largest group.

The problem of transformation of the given differential equation is also interesting and useful. If the admissible group is known it is possible with its help to search the transformation that would allow the transformed equation to have as far as it is possible the most simple and comfortable differential structure for finding the concrete solutions.

The enumerated and other more special problems form a large field of application of the theory of algorithms of continuous transformation groups. With the help of the theorems established by Sophus Lie it is possible to reduce the complicated non-linear problems to the linear ones that are simpler. This idea of “linearization” typical for analysis, but unusual for algebra, has got its realization in the creation of infinitesimal techniques of research. With this the local action of transformation group is changed on the action of linear differential operators – infinitesimal operators that form the Lie algebra. It is important that the passage from groups to Lie algebras fully reserves the algebraic structure of the studied equations. Therefore with its help it is possible to get the powerful infinitesimal criteria of invariance of various equations related to the continuous transformation group.

1.1. Local Transformation Groups

Let $K \subset R^r$ be an open sphere with the center at zero

$f : R^n \times K \to R^n$

are the smooth mappings, and T_a – the transformations R^n in itself defined by these mappings

$$T_a x = f(x,a), \quad x \in R^n, \quad a \in K, \tag{1}$$

$a = (a^1,...,a^r)$ are the parameters. It is supposed that $\forall\, a \in K,\ T_a$ are mutually unambiguous and $T_0 = E$ is an identical transformation and $T_a \neq E$ when $a \neq 0$.

The set of all the transformations (1) provided with the natural topology is called continuous r-parametric local group of transformations (further just transformation group) to R^n, if G_r is r–dimensional local Lie group relative to the operation

$$(T_b \cdot T_a)x = T_b(T_a x) \equiv f(f(x,a),b).$$

The unity in G_r is T_0. The product $T_b \cdot T_a$, and the inverse of T_a transformation T_a^{-1}, are defined $\forall\, a,b$ from the opened sphere $K' \subset K$ and with that the reflection $\varphi : K' \times K' \to K$ defined by group operation in G_r with the formula $T_b \cdot T_a = T_{\varphi(a,b)}$ is analytic. The value of the parameter corresponding to the inverse to T_a is denoted as a^-

^{1}, so that $T_a^{-1} = T_{a^{-1}}$.

The set $T_a x \ \forall\, x \in R^n$ is an orbit or the G_r orbit of point x. The orbit $A \subset R^n$ is the variety $G_r(A) = \bigcup_{x \in A} G_r(x)$.

1.2. Lie Equations

If K is an opened interval on the straight line R^1 then the orbit represents a curved line $a \to f(x,a)$ in R^n, passing through *x*. The tangential vector at point *x* has the form

$$\xi(x) = \left.\frac{\partial f(x,a)}{\partial a}\right|_{a=0}. \tag{2}$$

The vector field (2) of monoparametric group G_1 is written as the linear differential operator of the first order

$$X = \xi^i(x)\partial_{x^i} \equiv \sum_{i=1}^{n} \xi^i(x)\frac{\partial}{\partial x^i}\ , \tag{3}$$

called *infinitesimal operator* (or just *operator*) of group G_1.

The orbit of group $G_1(x)$ is an integral curved line of Lie equation

$$\frac{df}{da} = \xi(f), \quad f\big|_{a=0} = x. \tag{4}$$

Inversely for any smooth field $\xi : R^n \to R^n$ and $\forall\, x \in R^n$ there is the only monoparametric group G_1 of transformations $T_a x = f(x,a)$.

For r–parametric group G_r the tangential of the field for every r form an r–dimensional vector space L_r. It is a Lie algebra relative to multiplication

$$[\xi,\eta] = \eta'\xi - \xi'\eta$$

where ξ', η' are the derivatives of the mappings ξ,η. For the operators X, Y of the kind (3) the law of multiplication in L_r algebra is the following

$$[X,Y] = \left(X(\eta^i) - Y(\xi^i)\right)\frac{\partial}{\partial x^i}\ .$$

As the basis for L_r the following vector fields are chosen

$$\xi_\nu = \left.\frac{\partial f(x,a)}{\partial a^\nu}\right|_{a=0}, \qquad \nu = 1,\ldots,r. \tag{5}$$

The Lie equation looks like an integrable system

$$\frac{\partial f}{\partial a^{\nu}} = V_{\nu}^{\mu}(a)\xi_{\mu}(f), \quad \nu = 1,\ldots,r; \quad f\big|_{a=0} = x, \tag{6}$$

where the coefficients $V_{\nu}^{\mu}(a)$ of system (6) are defined according to the law of multiplication $\varphi(a,b)$ in the group G_r as

$$V_{\nu}^{\mu}(a) = \frac{\partial \varphi^{\mu}(a,b)}{\partial a^{\nu}}\bigg|_{b=a^{-1}}.$$

1.4. Invariants

The function $F(x)$ is an invariant of G_r group of transformations (1) to R^n if F is constant on G_r orbit of every point $x \in R^n$: $F(f(x,a)) = F(x)$.

The function $F : R^n \to R^n$ is an invariant of monoparametric group G_1 if and only if (iff)

$$XF \equiv \xi^{i}(x)\,\frac{\partial F(x)}{\partial x^{i}} = 0. \tag{7}$$

Any set $n-1$ of functionally independent solutions $J_1(x),\ldots,J_{n-1}(x)$ of equation (7) form the basis of invariants: any invariant F is presented as $F(x) = \Phi(J_1(x),\ldots,J_{n-1}(x))$.

For G_r the invariance criterion is the following:

$$\xi_{\nu}^{i}(x)\,\frac{\partial F(x)}{\partial x^{i}} = 0, \quad \nu = 1,\ldots,r, \tag{8}$$

where ξ_{ν} are the basic vector fields (5) of the Lie algebra L_r corresponding to G_r group. The number of solutions is defined by the value

$$r_*(\xi) = \operatorname{rank}\left[\xi_{\nu}^{i}(x)\right], \tag{9}$$

and the system (8) just possesses $n - r_*$, the invariants of G_r groups; when $r_* = n$ the group G_r has no invariants – it is transitive.

In more common situation when instead of R^n transformations the transformation groups are observed in arbitrary Banach space the role of the basis of invariants is played by the universal invariant of groups. Let B, B_1, B_2 be Banach spaces, G – a group of transformations to B and a reflection $J : B \to B_1$ is an invariant of group G, so $J_0 T = J \;\; \forall T \in G$. The invariant J is called *universal invariant* of group G, if for an arbitrary space B_2 every invariant is $F : B \to B_2$ of group G may be introduced as

$F = \Phi \circ J$ with the smooth reflection $\Phi : B_1 \to B_2$. The any r-parametric transformation group in the Banach space B has a universal invariant.

1.4. Invariant Manifolds

Let m –dimensional local variety $M \subset R^n$ is parameterized with the help of differential mapping $h : V \to R^n$, where V is an opened set of R^m, and the rank $h'(y) = m \quad \forall\, y \in V$. Tangential space to M in point $x \in M$ can be introduced as $M_x = \{dx \in R_x^n \mid dx = h'(y)dy,\ dy \in R_y^m\}$, where R_x^n is a tangential space to R^n in point x.

The variety M is an invariant relative to group G if M contains the G-orbit of every point $x \in M$, i.e. $G(M) = M$. The invariance criterion: $M \subset R^n$ is invariant relative to group G_r, when and only when for all the vector fields ξ from Lie algebra L_r of group G_r the condition

$$\xi(x) \in M_x$$

is observed at every point $x \in M$.

For $M \subset R^n$ set by the equation

$$F(x) = 0 \tag{10}$$

where $F : R^n \to R^{n-m}$, rank $F'(x) = n - m \ \forall\, x \in M$, the invariance criterion (8) looks like

$$\xi_\nu^i(x) \frac{\partial F(x)}{\partial x^i} \bigg|_M = 0, \quad \nu = 1,\ldots,r.$$

The variety M is not special relative to group G_r, if $r_*\left(\xi|_M\right) = r_*(\xi)$. When $r_*(\xi) < n$ the invariant relative to G_r is written as

$$\Phi^k\left(J_1,\ldots,J_{n-r_*}\right) = 0, \quad k = 1,\ldots,n-m, \tag{11}$$

where $J_1,\ldots,J_{n-r_*}$ is the basis of invariants of group G_r and the number

$$\rho = m - r_* \tag{12}$$

is called *the rank M.*

For an arbitrary $M \subset R^n$ the orbit $G_r(M)$ is the minimum invariant G_r that means the variety containing M as subvariety of codimension

$$\delta = \dim\ G_r(M) - \dim\ M\ .$$

The number δ is the *defect* M relative to group G_r. The invariant varieties are characterized by the condition $\delta = 0$. The defect of variety M, set by the equation (10) is equal to $\delta = \operatorname{rank}\left[\xi_\nu^i(x)\dfrac{\partial F^k(x)}{\partial x^i}\right]\Bigg|_M$.

The rank of the m-dimensional variety with the defect δ is

$$\rho = m - r_* + \delta, \tag{13}$$

and

$$\max\ (r_* - m; 0) \le \delta \le \min\ (r_* - 1; n - m - 1). \tag{14}$$

2. Invariant Differential Equations

2.1. The Continuation of Point Transformations

Let G_1 be monoparametric group of transformations

$$x' = f(x,u,a), \quad f\big|_{a=0} = x,$$

$$u' = \varphi(x,u,a), \quad \varphi\,\big|_{a=0} = u, \tag{15}$$

$x = \left(x^1,\dots,x^n\right)$, $u = \left(u^1,\dots,u^m\right)$, and $\underset{1}{u} = \left\{u_i^\alpha \,\middle|\, \alpha = 1,\dots,m;\ \ i = 1,\dots,n\right\}$, and the following transformations are given

$$u'^\alpha_i = \psi_i^\alpha\left(x,u,\underset{1}{u},a\right), \quad \varphi_i^\alpha\Big|_{a=0} = u_i^\alpha, \tag{16}$$

and the following conditions of agreement be observed

$$u_i^\alpha = \frac{\partial u^\alpha(x)}{\partial x^i}. \tag{17}$$

The monoparametric transformation group G_1 (15), (16) appears in the space R^{n+m+nm} of the variables $\left(x.u.\underset{1}{u}\right)$; (15) are *the point transformations*, (16) – *the continuation* of point transformations and $\underset{1}{G_1}$ *the first continuation* of group G_1. If

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{18}$$

is an operator of group G_1, where $\xi = \partial f / \partial u^\alpha |_{a=0}$, $\eta = \partial \varphi / \partial a |_{a=0}$, then the operator of continued group $\underset{1}{G_1}$ is equal to

$$\underset{1}{X} = X + \varsigma_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \tag{19}$$

$$\varsigma_i^\alpha = D_i\left(\eta^\alpha\right) - u_j^\alpha D_i\left(\xi^j\right),$$

and $D_i = \partial/\partial x^i + u_i^\alpha \, \partial/\partial u^\alpha$ is an operator fully differentiating with respect to the variable x^i.

The continuation to the higher orders is made by definition of the action of group G_1 on the variables $\underset{1}{u}, \underset{2}{u}, \ldots$

$$\underset{s}{u} = \left\{ u^\alpha_{i_1 \ldots i_s} \middle| \alpha = 1,\ldots,m;\ i_1,\ldots,i_s = 1,\ldots,n \right\}.$$

So the often used second continuation of operator (18)

$$X_2 = \underset{1}{X} + \varsigma_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha}$$

of group G is obtained by the continuation of the operator (19)

$$\varsigma_{ij}^\alpha = D_j\left(\eta_i^\alpha\right) - u_{ik}^\alpha \, D_j\left(\xi^k\right), \quad D_i = \frac{\partial}{\partial x^l} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha}.$$

The continuation of the arbitrary *r*-parametric transformation group and Lie algebra is got in an analogous way, taking into account the equality: if $Z = [X, Y]$, then $\underset{s}{Z} = \left[\underset{s}{X}, \underset{s}{Y}\right]$.

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Biographical Sketch

V.K. Andreev was born in village Kalinino of Nerchinsk region, Russia. He completed his Diploma in Mechanics and Applied mathematics at the Novosibirsk State University in 1972. In 1975 he took the Russian degree of Candidate in Physics and Mathematics at the Institute of Mathematics of the USSR Academy of Sciences, Novosibirsk. In 1990 V.K. Andreev was awarded the Russian degree of Doctor in Physics and Mathematics from Institute of Hydrodynamics of the USSR Academy of Sciences, Novosibirsk with the thesis “Invariant solutions of the hydrodynamics equations with a free boundary and their stability”. In 1992 he took Professor Diploma at the Chair “Mathematical analysis and differential equations”. From 1975 to 1990 he is a senior worker at the Computer Center of the USSR Academy of Sciences in Krasnoyarsk, Russia (now renamed the Institute of Computational Modeling of the Russian Academy of Sciences). Since 1990 V.K. Andreev is Head of Department “Differential equations of mechanics” at the same institute. V.K. Andreev is the well-known specialist in the fields of computational hydrodynamics, group analysis and hydrodynamic stability. He is author of five monographs and over 170 scientific papers, devoted mathematical modeling, numerical methods of solving boundary value problems, group classification and hydrodynamic stability.