FINITE ELEMENT METHOD

Jacques-Hervé SAIAC
Départment de Mathématiques, Conservatoire National des Arts et Métiers, Pairs, France

Keywords: Variational equations, energy minimization, Galerkin method, LaxMilgram's theorem, finite element meshes, Lagrangian interpolation, triangular P_k and quadrangular Q_k finite elements, isoparametric elements, quadrature formulas, error analysis.

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Summary

The introduction to finite element method may be very tricky. On the one hand, the rigorous foundation of the mathematical concepts requires difficult theorems of functional analysis, on the other hand, applications (to fluids for example) need sometimes complex finite element spaces. In the 3d case, with a great number of unknowns, it is very difficult to produce efficient codes. In this presentation, we prefer to concentrate on the fundamental concepts and to illustrate them by simple applications. We first present the finite element method in the one dimensional case, even if it is, of course, more useful in higher dimensions and complex geometry. This allows us to thoroughly explain the computations of element stiffness and mass matrices and complete linear systems with different boundary conditions. Then, we present two-dimensional triangular and iso-parametric quadrilateral elements and we give a general idea of the tridimensional case. Numerical integration formulas are detailed in the triangular and quadrilatal cases. A quick survey on error estimation and counter examples of bad choices of quadrature formulas are finally given.

1. Introduction

The Finite Element Method (F.E.M.) was developed in the late 1950's to numerically solve equations of elasticity and structural mechanics. It was introduced by engineers as a generalization of earlier methods used to solve discrete systems. The finite element method was based on an analogy between real discrete elements of a structure and small parts of a continuum domain, so-called finite elements. In the field of solid mechanics, it is nowadays the standard method. Outside this field, the range of applications of the finite method has extended to all engineering disciplines. Now this method is applied widely.

It has been developed by mathematicians and numerical analysts and has become a general tool for numerical solution of partial differential equations. The finite element method is particularly suitable to solve equilibrium or, equivalently, energy minimization problems. This method has the big advantage that complicated geometries, general boundary conditions and variable material properties can be handed easily and in a natural way, whereas finite differences or spectral methods introduce artificial complications. In fluid mechanics problems, the finite element method competes with the finite volume method which presents the same geometrical flexibility and naturally preserves the conservation laws. Modern codes combine both methods, finite volume dealing with the conservative part of the equations and finite element with the dissipative one. For a mechanical engineering approach and a review of many applications of the finite element method, refer to [8]. For a mathematical study and a detailed error analysis, refer to [1]. For teaching textbooks, refer to [5] and [6]. At last for finite element applications to Naveir-Stokes equations and fluid mechanics, the reader will profitably read [4], [7] and [3] for extensions to non-linear problems.

1.4. A Simple One-dimensional Problem

The variational formulation of elliptic differential problems and the equivalent energy minimization problem (in the symmetric case) are the key basis of finite element methods (see *Variational statements of problems. Variational methods*). Once the continuous problem is written in variational form, thanks to the Lax Milgram theorem, we obtain the existence and uniqueness of the exact solution in Hilbert space *H*. Then, the approximation process begins. The finite element method enables us to build finite dimensional spaces (subspaces of *H* for the so-called conforming methods).The approximate solution belongs to this finite dimensional space. It is, in a certain sense, the best approximation in this subspace of the exact solution. In contrast with finite difference methods where the approximated solution is the vector of discrete nodal approximate values, in the finite element method, the approximate solution is a continuous (in most cases) function (even if it is, finally, defined by its nodal values). This is one of the big advantages of finite element methods. We can apply to the approximate solution the same operators as we applied to the exact solution.

Let us begin by a simple one-dimensional problem to introduce the basic concepts of the finite element method (F.E.M). An elastic cord fixed at both ends is subject to a vertical load per unit length *f*. The vertical displacement *u* is real function, the solution of

$$\begin{cases} -u''(x) = f(x) & \forall x \in [0, L] \\ u(0) = 0 & u(L) = 0 \end{cases} \tag{0.1}$$

where L is the length of the cord.
This very simple homogenous Dirichlet problem can be written in variational form. We proceed as in *Variational statements of problems. Variational methods*. to obtain

$$\begin{cases} \text{Find } u \in V_0 \quad \text{such that} \quad \forall v \in V_0 \\ \mathcal{A}(u,v) = L(v) \end{cases} \tag{0.1}$$

where the space $V_0 = H_0^1(\Omega)$, with $\Omega = [0, L]$, and the bilinear form $\mathcal{A}$ and the linear form L are defined by:

$$\mathcal{A}(u,v) = \int_0^L \frac{du}{dx}(x)\frac{dv}{dx}(x)\,dx, \quad L(v) = \int_0^L f(x)v(x)\,dx \tag{0.2}$$

As the bilinear form is symmetric, the problem is equivalent to the following energy minimization:

$$\begin{cases} \text{Find} \quad u \in V_0 \text{ which minimizes} \\ E(v) = \dfrac{1}{2}\displaystyle\int_0^L \left(\frac{dv}{dx}(x)\right)^2 dx - \int_0^L f(x)v(x)\,dx \end{cases} \tag{0.3}$$

1.5. Approximation Process with Linear Elements

This is the starting point of the approximation process. Then F.E.M. can be simply described as a process of constructing finite dimensional subspaces $V_{0,h}$ of V_0. The approximate solution will be found in five steps.

1. Build a mesh on the domain Ω. The domain Ω is subdivided into a finite number of simple subsets, called finite elements. In this first one-dimensional case, elements are the N subintervals $[x_{i-1}, x_i]$ for $i = 1, \ldots N$.

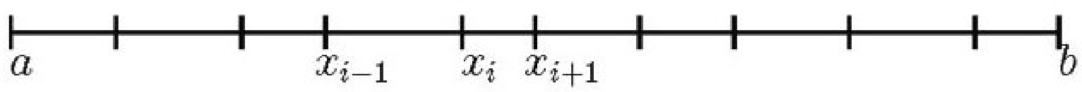

Figure: 1. One dimensional mesh

2. Choose a simple function space to represent the approximate solution locally on each element. Here we will choose linear polynomials. Then define the global finite dimensional subspace $V_{0,h}$ based on the mesh by considering the continuous functions satisfying the Dirichlet boundary conditions and whose restrictions on each element are linear polynomials. Approximate solutions of the following kind are obtained:

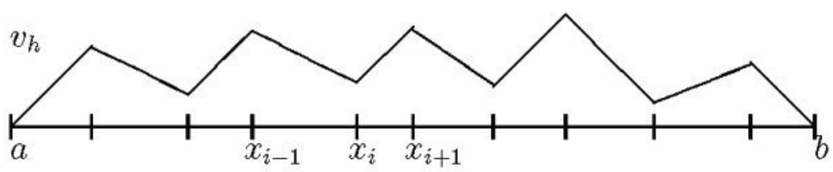

Figure: 2. A function of the discrete space $V_{0,h}$

A Lagrangian basis $\{w_i\}$ of this space is the set of “hat” functions defined by

$$w_i\left(x_j\right)=\delta_{ij} \quad \forall i=1,N-1 \quad \text{and} \quad \forall j=1,N-1 \tag{0.4}$$

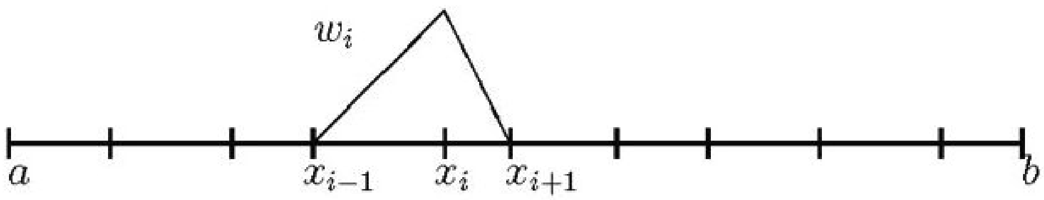

Figure:3. A basis function

Any function $v_h \in V_{0,h}$ can be represented by:

$$v_h(x)=\sum_{i=1}^{i=N-1} v_i w_i(x) \tag{0.5}$$

where $v_i = v_h(x_i)$. The coefficients v_i are nothing but the nodal values of v_h.

3. Project the continuous problem on the finite dimensional subspace $V_{0,h}$. The following discrete problem will be solved:

$$\begin{cases} \text{Find } u_h \quad \in V_{0,h} \quad \text{such that} \quad \forall v_h \in V_{0,h} \\ \mathcal{A}\left(u_h, v_h\right)=L\left(v_h\right) \end{cases} \tag{0.6}$$

This problem is equivalent to the linear system:

$$\begin{cases} \text{Find } u_1, u_2, \ldots u_{N-1} \quad \text{such that } \forall i=1,N-1 \\ \displaystyle\sum_{j=1}^{j=N-1} \mathcal{A}\left(w_j, w_i\right) u_j = L\left(w_i\right) \end{cases} \tag{0.7}$$

4. Compute the matrix and the right hand side coefficients of this linear system. This is done by splitting the integrals into contributions from each element. On each element (here intervals $\left[x_{i-1}, x_i\right]$), the basis functions w_i reduce to linear polynomials. That makes the computations very easy.

After elementary calculations are performed, the resulting linear system is obtained by an assembling process.

5. Finally the linear system is solved using one of the methods described in *Solution of systems of linear algebraic equations.*

1.6. Computation of Matrix Coefficients

Let us give some details on the practical computation of matrix coefficients in this simple case. The computation is done by assembling the contributions of each element

$T_i = \left[x_{i-1}, x_i\right]$ for $i = 1, \ldots N$.

The coefficients

$$A_{ij} = \mathcal{A}\left(w_j, w_i\right) = \int_0^L w_j'(x) w_i'(x)\,dx$$

of the global matrix A are obtained by adding elementary contributions:

$$A_{ij} = \sum_{k=1}^{k=N} \int_{x_{k-1}}^{x_k} w_j'(x) w_i'(x)\,dx \,. \tag{0.8}$$

Let us consider, for example the element $T_i, = \left[x_{i-1}, x_i\right]$. There are only two non zero basis functions on T_i, namely w_{i-1} and w_i

$$w_{i-1}\big|_{T_i} = \frac{x_i - x}{x_i - x_{i-1}} \quad w_i\big|_{T_i} = \frac{x - x_{i-1}}{x_i - x_{i-1}} \tag{0.9}$$

$$w'_{i-1}\big|_{T_i} = \frac{-1}{x_i - x_{i-1}} \quad w'_i\big|_{T_i} = \frac{1}{x_i - x_{i-1}} \tag{0.10}$$

Thus the contribution of T_i will affect only the four following global matrix coefficients: $A_{i-1,i-1}, A_{i-1,i}, A_{i,i}$ and $A_{i,i-1}$.

It is easy to compute these contributions. They are usually written in a matrix form, leading to the so called "*element matrix*":

$$Elem_i = \begin{pmatrix} e_{1,1}^i & e_{1,2}^i \\ e_{2,1}^i & e_{2,2}^i \end{pmatrix} \tag{0.11}$$

where

$$e_{1,1}^i = \int_{x_{i-1}}^{x_i} w'_{i-1}(x)^2\,dx = \int_{x_{i-1}}^{x_i} \frac{1}{\left(x_i - x_{i-1}\right)^2}\,dx = \frac{1}{x_i - x_{i-1}} \tag{0.12}$$

$$e_{1,2}^i = e_{2,1}^i = \int_{x_{i-1}}^{x_i} w_{i-1}'(x) w_i'(x) dx = \int_{x_{i-1}}^{x_i} -\frac{1}{(x_i - x_{i-1})^2} dx = -\frac{1}{x_i - x_{i-1}} \qquad (0.13)$$

$$e_{2,2}^i = \int_{x_{i-1}}^{x_i} w_{i-1}'(x)^2 dx = \int_{x_{i-1}}^{x_i} \frac{1}{(x_i - x_{i-1})^2} dx = \frac{1}{x_i - x_{i-1}} \qquad (0.14)$$

and then

$$Elem_i = \begin{pmatrix} \frac{1}{x_i - x_{i-1}} & \frac{-1}{x_i - x_{i-1}} \\ \frac{-1}{x_i - x_{i-1}} & \frac{1}{x_i - x_{i-1}} \end{pmatrix} = \frac{1}{x_i - x_{i-1}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad (0.15)$$

The same kind of computational technique is used to obtain the right hand side of the resulting linear system.

2. Other One-dimensional Boundary Problems

We can apply the same process to more general one-dimensional boundary value problems.

2.1. The Non-homogeneous Dirichlet Problem

Let us consider the following Dirichlet problem:

$$\begin{cases} -u''(x) = f(x) & \forall x \in [0, L] \\ u(0) = \alpha \quad u(L) = \beta \end{cases} \qquad (2.1)$$

We introduce an auxiliary function u_0 such that

$$u_0(0) = \alpha, \quad u_0(L) = \beta,$$

and we consider the new unknown function $\tilde{u} = u - u_0$. Problem (2.1) is then equivalent to a homogeneous problem for $\tilde{u}$. A practical choice for u_0 consists in expanding it in the finite element basis

$$u_0 = \alpha w_0 + \beta w_N,$$

where w_0 and w_N are the P1-Lagrangian basis functions associated with the boundary points $x_0 =$ and $x_N = L$. Thus, $\tilde{u}$ is solution of

$$\begin{cases} \text{Find } \tilde{u} \in V_0 \quad \text{such that} \quad \forall v \in V_0 \\ \mathcal{A}(\tilde{u}, v) = L(v) \end{cases} \qquad (2.2)$$

where the bilinear form $\mathcal{A}$ is the same as in the homogeneous case, but the linear form L is now equal to

$$L(v) = \int_0^L f(x)v(x)\,dx - \mathcal{A}(u_0, v). \qquad (2.3)$$

Then, the approximation process follows the same lines.

2.2. The Neumann Problem

We consider now the Neumann problem $\left(c \in L^\infty(0,L)\right)$:

$$\begin{cases} -u''(x) + c(x)u(x) = f(x), & x \in [0,L], \\ u'(0) = \alpha \quad u'(L) = \beta \end{cases} \qquad (2.4)$$

The variational formulation of this problem is:

$$\begin{cases} \text{Find } u \in V \quad \text{such that} \quad \forall v \in V \\ \mathcal{A}(u,v) = L(v) \end{cases} \qquad (2.5)$$

where, this time, the space $V = H^1(\Omega)$, with $\Omega = [0,L]$, and the bilinear form $\mathcal{A}$ and the linear form L are defined by:

$$\mathcal{A}(u,v) = \int_0^L \frac{du}{dx}(x)\frac{dv}{dx}(x)\,dx, \quad L(v) = \int_0^L f(x)v(x)\,dx + \beta v(L) - \alpha v(0) \qquad (2.6)$$

Remark 2.1 (see *Variational statements of problems. Variational methods***).** If $c \equiv 0$, problem (2.4) is the Neumann problem for the Laplace operator: it is an "ill-posed" problem.

Following the same approach as before, the finite dimensional space $V_h \subset V$ is the space of continuous, piecewise linear, functions. But, this time, there is no restriction on boundary values. The dimension of the discrete space V_h is equal to $N+1$: the number of nodes of the mesh. The approximate solution reads as follows

$$u_h(x) = \sum_{i=0}^{i=N} u_i w_i(x). \qquad (2.7)$$

The discrete problem to solve is:

$$\begin{cases} \text{Find } u_h \in V_h \quad \text{such that} \quad \forall v_h \in V_h \\ \mathcal{A}(u_h, v_h) = L(v_h) \end{cases}. \qquad (2.8)$$

Practical computations follow the same lines as above.

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Bibliography

[1] Ciarlet P.G., *The Finite Element Method for Elliptic Problems*, North Holland (1979) [A reference on finite element analysis.].

[2] George P.L., *Automatic mesh generation. Applications to finite element methods*, J. Wiley and Sons (1991). [A key reference on automatic meshing.].

[3] Glowinski R., *Numerical Methods for Nonlinear Variational Problems*, Springer Verlag (1984) [A very useful book on nonlinear problems and industrial applications. The appendix on linear variational problems is one of the best introduction of variational method and finite element approximation ever written.].

[4] Giralut V., Raviart P.A., *Finite Element Methods for Navier-Stokes Equations, Theory and Applications*, Springer Verlag (1986) [The reference on finite element analysis for Navier-Stokes equations.].

[5] Johnson C., *Numerical solutions of Partial Differential Equations by the Finite Element Method*, Cambridge University Press (1987) [A very pedagogical textbook on finite element concepts].

[6] Lucquin B., Pironneau O. *Introduction to scientific computing*, J. Wiley and Sons (1998) [An easy reading but rigorous and exhaustive textbook on numerical methods for solving partial differential equations].

[7] Pironneau O., *Finite Elements for Fluids*, J. Wiley and Sons (1989) [The reference on finite element application to fluids].

[8] Zienkiewicz O.C., Taylor R.L. *The Finite Element Method, fourth edition*, Mac Graw Hill (1987) [A well-known reference on finite element method and applications].

Biographical Sketch

Jacques-Hervé Saiac is Professor at the Conservatoire National des Arts et Métiers (CNAM) in Paris, France. He graduated from the Ecole Centrale de Paris in 1968 and received his Ph. D. from UPMC Paris VI University in 1971. In 1991, under the direction of Professor Olivier Pironneau, he received the Habilitation à diriger des recherches (HDR) diploma in Applied Mathematics from UPMC Paris VI University. His research fields include finite element analysis, optimization methods, numerical simulations in Computational Fluid Dynamics and adaptive mesh analysis.