INVERSE PROBLEMS AND IMAGING: PAST, PRESENT AND FUTURE

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Keywords: Inverse problems, backwards heat equation, computerized tomography, inverse scattering, ill-posed problems, stability, transmission eigenvalues.

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Summary

Inverse problems focus on the problem of determining parameters and data inherent in the mathematical model of a physical or biological phenomenon from measurements of the observable data. Such problems are almost invariably ill-posed in the sense that in general existence, uniqueness or continuous dependence on the data is no longer true. In this chapter we have chosen three “canonical” examples of inverse problems, have described their physical origin and then present mathematical methods which in each case address the basic issues of existence, uniqueness and stability for what are fundamentally ill-posed problems. These examples are the backwards heat equation, computerized tomography and the inverse scattering problem for acoustic waves. To aid the reader in our discussion we have also presented a brief introduction to the theory of Hilbert spaces where we have only supposed a pre-requisite of elementary linear algebra and calculus.

1. Introduction

Although not recognized as a mathematical discipline until recently, inverse problems are as old as science itself. In particular a working definition of science is the problem of constructing a model of some physical or biological phenomena that, although inexact, is accurate enough to be able to use observations or measurements to obtain information about the phenomena under investigation. Such models are typically mathematical in nature and hence the challenge is to “invert" the model to recover useful estimates of the object under investigation. Since the model itself is inexact, such an inversion process typically leads to problems of existence and stability.

Strangely enough, given the above description of the scientific method, the mathematical theory of inverse problems was essentially ignored until the middle of the twentieth century. Instead, scientists focused on *direct problems*, i.e. the construction of the model itself rather than the inversion process. In particular, direct problems are based on developing a mathematical model that maps causes into effects and are typically *well-posed*: each cause has a unique effect and causes which are close to one another have effects which are close to each other. The scientific phenomena were then investigated by adjusting the input to the direct problem such that the output fit the measured data. By the beginning of the twentieth century, the idea of direct problems dominated mathematical physics. Indeed, the French mathematician Hadamard held the opinion that an important physical problem *must* be well-posed, i.e. the problem must always have a unique solution that depends continuously on the data.

The attitude typified by that of Hadamard persisted well into the middle of the twentieth century. However the advent of quantum mechanics and numerous problems in areas of classical physics such as heat conduction and geophysics soon slowly convinced mathematicians and scientists that well-posed direct problems were not the only ones of scientific interest and, pioneered by mathematicians in the Soviet Union led by Tikhonov, the mathematical theory of *inverse problems* began to be developed. In particular, this theory focused on the problem of determining the parameters and data in the mathematical model of the direct problem from measurements and observations of the data that arise from the physical or biological phenomena taking place. Such problems are almost invariably *ill-posed* in the sense that in general either existence, uniqueness or continuous dependence on data is no longer true. Although the problems of existence and uniqueness in inverse problems can often be ameliorated by generalizing the notion of solution and constructing a generalized solution, the key attribute of stability is often absent in inverse problems unless further a priori information is available. This essential lack of stability usually has dire consequences when numerical methods using measured (and hence inexact) data are applied to inverse problems.

In view of the inherent problem of instability that is characteristic of inverse problems, the mathematical theory of inverse problems focuses on this issue. In particular, the primary problem that needs to be addressed is what type of a priori information is "normally" available and how can this information be brought into the mathematical model? In this context, the solution space and the space of observations are typically taken to be Hilbert spaces (but not necessarily the same Hilbert space since one desires more of the solution than one demands from the observation). The mathematical model itself is then an operator taking one Hilbert space into another, i.e. the mathematical foundation of inverse problems is the theory of operators in a Hilbert space. Until recently, most of the mathematical theory of inverse problems was focused on linear problems and hence the theory of linear operators in a Hilbert space. However, in recent years, more and more attention has been focused on nonlinear problems where, in addition to stability, uniqueness issues are seen to play a prominent role.

The purpose of this chapter is to give a brief survey of the field of inverse problems. However this is by no means an easy task since the field has experienced tremendous growth in the past fifty years covering areas as diverse as computerized tomography, synthetic aperture radar, geophysical prospecting and nondestructive testing. Since the solution of any inverse problem is to "invert" the model to recover useful information about the physical phenomena from the observed image, inverse problems by definition must also deal with the subject of imaging. A comprehensive survey of many areas of inverse problems and imaging can be found in recently published 1600 page handbook [28]. We will make no attempt to survey in twenty pages that which can only be partially done in 1600 pages. Instead we have chosen three "canonical" examples of inverse problems, have described their physical origin and then presented mathematical methods which in each case address the basic issues of existence, uniqueness and stability for what are fundamentally ill-posed problems. The mathematical methods that we present are of course also applicable to many other inverse problems which are not discussed here but can be found in the above mentioned handbook. For the mathematically unsophisticated reader we have also presented a brief introduction to theory of Hilbert spaces where we have only supposed a pre-requisite of elementary linear algebra and calculus. We conclude our chapter by presenting a subjective attempt to see into the future of inverse problems. Here, among many possible choices, we have chosen the area of obtaining inequalities of physical interest in scattering theory from a knowledge of the measured scattered wave. Such techniques provide a rapid method to obtain valuable information from what is basically a complicated multi-dimensional nonlinear inverse scattering problem with many possible future applications in nondestructive testing. Only the future will tell if this direction will in fact bear fruit!

2. Three Examples of Inverse Problems

We will now give three examples of inverse problems which will serve as our model problems in what follows. The first example is the backwards heat equation which is perhaps the simplest model of a linear inverse problem and was one of the first ill-posed problems that was systematically studied (c.f. [27]). The second example is again a linear inverse problem, but this time one that is considerably more difficult to analyze. This is the problem of computerized tomography which has revolutionized medical imaging and for which its inventors won the Nobel Prize for medicine. Our third and final example is the inverse scattering problem for acoustic waves which is the best known example of a nonlinear inverse problem and, in its electromagnetic version, is the mathematical basis of synthetic aperture radar (c.f. [7]).

2.1. The Backwards Heat Equation

Before presenting the ill-posed problem of solving the backwards heat equation, we note that in order to investigate a problem that is ill-posed we must answer two basic questions: 1) What do we mean by a solution? and 2) How do we construct this solution? The answers to these questions are by no means trivial. For example, as initially posed a solution may not even exist in the classical sense. In this context it is worthwhile recalling a remark of Lanczos: “A lack of information cannot be remedied by any mathematical trickery". Hence, in order to determine what we mean by a solution it is often necessary to introduce “nonstandard" a priori information gained from a knowledge of the physical situation that one is trying to model. Even after we have resolved the problem of what we mean by a solution, there remains the problem of actually constructing such a solution, and this is often complicated by the fact that the above mentioned nonstandard information has been incorporated into the mathematical model, thus leading to nonstandard problems in analysis.

We now turn our attention to the backwards heat equation. Physically, the problem that we are about to consider is to determine the temperature of a solid in the past from a knowledge of its temperature in the present and the temperature on the boundary of the solid in the past. Mathematically, we can formulate this problem in the following manner (assuming zero boundary data and a homogeneous medium): Find a solution u of

$$\Delta_3 u = u_t \qquad \text{in } D \times [0,T] \tag{1}$$

$$u = 0 \qquad \text{on } \partial D \times [0,T] \tag{2}$$

$$u(x,T) = f(x) \qquad \text{for } x \in D \tag{3}$$

for a prescribed function f, where D denotes the given solid. It can be shown that no solution exists to this problem unless f is an analytic function of its three independent variable. Furthermore, even if f is analytic the solution, if it exists, does not depend continuously on the data f. To see this let φ_n be an (orthonormalized) eigenfunction corresponding to an eigenvalue λ_n of

$$\Delta_3 \varphi + \lambda \varphi = 0 \qquad \text{in } D \tag{4}$$

$$u = 0 \qquad \text{on } \partial D. \tag{5}$$

Then

$$u_n(x,t) = \frac{1}{\lambda_n} \varphi_n(x) e^{-\lambda_n (t-T)}$$

is a solution (1)-(3) for

$$f(x) = f_n(x) = \frac{1}{\lambda_n} \varphi_n(x)$$

and since $\|\varphi\| = 1$ where $\|\cdot\|$ is the L^2-norm over D, we have that $\|f_n\| = 1/\lambda_n$. But for each fixed t, $0 \le t < T$, we have that

$$\|u_n(x,t)\| = \frac{1}{\lambda_n} e^{-\lambda_n (t-T)}$$

and since $\lambda_n \to \infty$ as $n \to \infty$ [8] we have that $\|f_n\| \to 0$ as $n \to \infty$ whereas $\|u_n(x,t)\| \to \infty$. Thus the solution of (1)-(3) does not depend continuously on the data f.

In Section 4 of this chapter we will show how the above problems can be avoided if we look for a solution of (1)-(3) in the class of solutions to (1)-(3) that satisfy an a priori bound and define a “solution” to be a function in this class that best approximates the data (3) in $L^2(D)$.

2.2. Computerized Tomography

Literally, tomography means *slice imagining*. Today this term is applied to many methods used to reconstruct the internal structure of a solid object from external measurements [17], [26]. We consider here a mathematical model of the measurement process used in transmission computerized tomography. Here, a cross-section of the human body is scanned by a thin *x*-ray beam whose intensity loss is recorded by a detector and processed by a computer to produce a two-dimensional image of slices of the human body which in turn are displayed on a screen. More specifically, objects of interest in *x*-ray imaging are described by a real-valued function defined on $\mathbb{R}^3$, called the *attenuation coefficient*, which quantifies the tendency of an object to absorb or scatter *x*-rays of a given energy. This function, denoted here by $\mu(x) \ge 0$ for $x \in \mathbb{R}^3$, varies from point-to-point within the object and is usually taken to vanish outside. Our model for the interaction of *x*-rays with matter is based on three basic assumptions: 1) *x*-rays travel along straight lines that are not "bent" by the object they pass through, 2) the waves making up the *x*-ray beam are all of the same frequency, and 3) the intensity *I* of the *x*-ray beam satisfies Beer’s law

$$\frac{dI}{ds} = -\mu(x) I \qquad (6)$$

where s is the arc-length along the straight-line trajectory of the *x*-ray beam. In a real measurement the total energy, I_i incident on the subject along a given line ℓ is given, and the total energy I_o emerging from the object along ℓ is measured by an *x*-ray detector. Hence, integrating (6) we obtain

$$\log \frac{I_\mathrm{o}}{I_\mathrm{i}} = \int_\ell \mu ds. \qquad (7)$$

By varying the position of the source we can measure the quantity on the left hand side of (7) along a family of lines [17]. In computerized tomography the function $\mu(x)$, $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, is reconstructed from its two-dimensional slices, i.e. $f_c(x_1,x_2):=\mu(x_1,x_2,c)$ for a given c. Now suppose that the support of $\mu(x)$ is inside the cube $[-a,a]\times[-a,a]\times[-a,a]$. For each fixed c between $\pm a$ and each pair $(t,\omega)\in\mathbb{R}\times S^1$, where S^1 is the unit circle, we measure the line integral of f_c along the line lying on the plane $x_3=c$

$$\left\{(x_1,x_2,x_3): x_3=c, \text{and } (x\cdot\omega)=t,\ x=(x_1,x_2)\right\},$$

that is,

$$\int_{-\infty}^{\infty} f_c\left(t\omega+s\omega^{\perp}\right)ds,$$

where $\omega(\theta):=(\cos(\theta),\sin(\theta))$, $\omega^{\perp}$, is the direction perpendicular to ω, i.e. $\omega^{\perp}(\theta):=(-\sin(\theta),\cos(\theta))$, and $(\cdot)$ denotes the $\mathbb{R}^2$ dot product. In this idealized model it is assumed that on the plane $x_3=c$ the sources and receivers are moved around a circle enclosing the corresponding slice of absorbent material of compact support, i.e. $0\le\theta<2\pi$.

The above measurement model in CT scanning brings us to an essential mathematical problem: *Can a two-variable function be recovered from a knowledge of its line integrals along all lines?* This leads to the definition of the *Radon Transform* which in the following is defined only in $\mathbb{R}^2$ (see [15], [26] for a discussion on the Radon transform in higher dimensions). To this end we identify $\mathbb{R}\times S^1$ with the space of oriented lines $\ell_{t,\omega}$ given by

$$\ell_{t,\omega}:=\left\{x\in\mathbb{R}^2:\langle\omega,x\rangle=t\right\}=\left\{t\omega+s\omega^{\perp}:S\in\mathbb{R}\right\} \tag{8}$$

Definition 2.1. *Suppose that f is a function defined in the plane which, for simplicity, we assume is continuous with bounded support. The integral of f along the line $\ell_{t,\omega}$ is denoted by*

$$\mathcal{R}f(t,\omega)=\int_{\ell_{t,\omega}} f ds=\int_{-\infty}^{\infty} f\left(t\omega+s\omega^{\perp}\right)ds.$$

The collection of all integrals of f along the line ℓ on the plane defines a function on $\mathbb{R}\times S^1$, called the Radon transform *of f.*

It is not necessary for f to be either continuous or of bounded support. The Radon transform can be defined for a function f whose restriction to each line is locally integrable and

$$\int_{-\infty}^{\infty} \left| f\left(t\omega + s\omega^{\perp}\right) ds < \infty, \right| \quad \text{for all} \left(t,\omega\right) \in \mathbb{R} \times S^1 \tag{9}$$

Thus, computerized x-ray tomography becomes the problem of inverting the Radon transform of slices $f_c\left(x_1, x_2\right) := \mu\left(x_1, x_2, c\right)$, $-a \le c \le a$, of the attenuation coefficient μ of the object of interest. We shall address this problem in Section 4 and Section 5, where mathematical questions such as uniqueness, stability and of course reconstructions methods are briefly discussed. Note that many other problems in tomography and imaging can be re-written as the problem of inverting the Radon transform of some function of interest (for more details see [15], [25] and [26]).

In practice the above integrals can be measured only for a finite number of lines using basically two scanning geometries, namely parallel scanning and fan-beam scanning. Thus the real problem in computerized tomography is to reconstruct a slice f_c from a finite number of its line integrals. Sometimes it is neither possible nor desirable to scan the whole cross-section. One then has to reconstruct f_c from the integrals corresponding to limited angle aperture, i.e. one speaks of the incomplete data problem. In particular, if a three-dimensional model is adapted in order to increase the efficiency of the procedure, the incomplete data problem is the rule. The line sampling and the angle aperture of course impact the accuracy and the resolution of the image. Finally, the model discussed here is highly idealized and in practice further model corrections are introduced to account for the width of the beam, the energy dependence of the attenuation etc. We refer the reader to [26] for a more detailed discussion on these issues.

2.3. The Inverse Scattering Problem

The propagation of time harmonic acoustic waves of frequency $\omega > 0$ through a homogeneous medium in $\mathbb{R}^3$ with speed of sound c is governed by the *Helmholtz equation*

$$\Delta_3 u + k^2 u = 0 \tag{10}$$

where the wave number $k = \omega/c$. A solution of the Helmholtz equation whose domain of definition contains the exterior of some sphere is called radiating if it satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{11}$$

where $r = |x|$ and the limit holds uniformly in all directions $\hat{x} = x/|x|$.

We will consider two basic problems in scattering theory, namely scattering by a bounded impenetrable obstacle and scattering by a penetrable inhomogeneous medium of compact support. For a vector $d \in \mathbb{R}^3$ with $|d| = 1$ the function $e^{ikx \cdot d}$ satisfies the Helmholtz equation for all $x \in \mathbb{R}^3$. It is called a *plane wave* since $e^{i(kx \cdot d - \omega t)}$ is constant on the plane $kx \cdot d - \omega t$ equal to a constant. Assume that an incident field is given by the plane wave $u^{\mathrm{i}}(x) = e^{ikx \cdot d}$. Then the simplest obstacle scattering problem is to find the scattered field u^{s} as a radiating solution to the Helmholtz equation in the exterior of a bounded scatterer D such that the total field

$$u = u^{\mathrm{i}} + u^{\mathrm{s}} \tag{12}$$

satisfies the Dirichlet boundary condition

$$u = 0 \qquad \text{on} \quad \partial D. \tag{13}$$

The simplest scattering problem for an inhomogeneous medium assumes that the speed of sound is constant outside a bounded domain D. Then, if u^{i} again is given by $u^{\mathrm{i}}(x) = e^{ikx \cdot d}$, the total field $u = u^{\mathrm{i}} + u^{\mathrm{s}}$ satisfies

$$\Delta_3 u + k^2 n u = 0 \qquad \text{in} \quad \mathbb{R}^3 \tag{14}$$

and the scattered field u^{s} fulfills the Sommerfeld radiation condition (11). Here the wave number k is given by $k = \omega / c_0$ and $n = c_0^2 / c^2$ is the *index of refraction* where c_0 is the sound speed in the homogeneous background medium and $c = c(x)$ is the speed of sound in the inhomogeneous medium. We define $n(x) = 1$ for $x \notin D$. An absorbing medium is modeled by adding an absorption term which leads to a refractive index with a positive imaginary part

$$n = \frac{c_0^2}{c^2} + i\frac{\gamma}{k}$$

where $\gamma = \gamma(x)$ is the absorbing coefficient.

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Biographical Sketches

Fioralba Cakoni was born in Elbasan, Albania on August 14, 1964. She graduated with B.S. and M.S. Degrees in Mathematics from the University of Tirana and received her PhD in Mathematics from the same University in 1996. She was an Alexander von Humboldt Research Fellow at the University of Stuttgart from 1998 to 2000. She then joined the faculty at the University of Delaware in 2000, becoming professor in 2010. Her mathematical areas of expertise includes integral equations and variational methods for elliptic partial differential equations, direct and inverse boundary value problems and direct and inverse scattering theory. Her main research interest is in the area of inverse scattering theory for acoustic, elastic and electromagnetic waves, in particular the use of qualitative methods to determine the shape and material properties of the scattering object. She is the author, with David Colton, of *Qualitative Methods in Inverse Scattering Theory* (Springer, 2006) and, with David Colton and Peter Monk, of *The Linear Sampling Method in Inverse Electromagnetic Scattering* (SIAM Publications, 2011). She has written over 50 peer-reviewed research articles and has given numerous invited talks throughout the world.

David Colton was born in San Francisco on March 14, 1943. He received hid B.S. Degree with honors in Mathematics from the California Institute of Technology in 1964, his M.S. degree from the University of Wisconsin in 1965 and his PhD from University of Edinburgh in 1964. In 1977 he was awarded the Doctor of Science degree from the University of Edinburgh. From 1967 to 1975 he was an Assistant and Associate Professor at Indiana University, from 1975 to 1978 he held the Chair of Applied Analysis at the University of Strathclyde in Glasgow, Scotland, and has been at the University of Delaware since

1978 where he was appointed UniDel Professor in 1996. His mathematical areas of expertise include analytic function theory, integral equations, asymptotic analysis and partial differential equations . His current research interests are in inverse scattering theory for acoustic and electromagnetic scattering theory. In addition to the above monographs with Fioralba Cakoni he is the author of *Analytic Theory of Partial Differential Equations* (Pitman, 1980), *Partial Differential Equations: An Introduction* (Random House 1988, Dover Publications 2004) , *Integral Equation Methods in Scattering Theory* (John Wiley, 1983) with Rainer Kress and *Inverse Acoustic and Electromagnetic Scattering Theory* (Springer 1992, Second Edition 1998, Third Edition, 2013) with Rainer Kress. He is the author of author of over 200 papers and has given numerous lectures throughout the world.