LINEAR PROGRAMMING

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Keywords: optimization, linear programming, simplex method, ellipsoid method, interior point methods, duality, worst-case complexity, average-case complexity, polyhedra, linear inequalities

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Summary

This is a survey of the various applications of Linear Programming to real-world decision making. It clarifies the essential mathematical terms and principal theoretical ideas in that mathematical field, and explains the most useful and efficient solution methods.

Linear Programming is a mathematical concept used to simulate well-known, real-world situations, in which one wants to achieve a certain goal, but has to consider certain constraints while searching for the best possible decision. In Linear Programming, one assumes that the decisions have an impact both on the objective and on the restrictions which can be characterized by linear functions of the variables characterizing the decisions. Then, it is the task of mathematics to calculate the optimal values of the decision variables. We present the mathematical theories of linear inequalities, of polyhedra and of duality. Then, we show how they can be exploited to develop algorithms for solving such problems technically. The three methods that are discussed

are the Simplex Method, the Ellipsoid Method and Interior Point Methods. Their description finally leads to a comparison of their relative efficiencies and complexities.

1. Linear Programming Problems

1.1. Formulation of Linear Programming Problems

In real life, most of our decisions about how to act or how to behave can be interpreted as attempts to optimize a certain goal or objective, without violating certain restrictions. The restrictions may be given by nature, people, our own will, or by existing rules.

For mathematics, the challenge is to modelize and formalize this process mathematically, and to provide calculation methods for the determination of the best possible decision, if the objective and the restrictions are known. This mathematical field of developing tools for that purpose is of enormous importance in economy, engineering, administration, communication, and in all questions concerning technological development.

The mathematical approach for solving such problems is as follows:

Translate the possible decision set into a formal set of decision vectors with a finite number of decision variables. Then determine which of these decision-vectors are feasible under the given restrictions. After that, optimize, i.e., select the best decision-vector from the feasible ones. The criterion for good, better and best comes from a mathematical function describing the specific quality of the decision in question.

A formal characterization of the mathematical problem is:

$$\begin{array}{ll} \text{maximize} & f(x), \quad \text{a function } f:\mathbb{R}^n \to \mathbb{R} \text{ defined on } x=(x^1,\ldots,x^n)^T \\ \text{subject to} & g_1(x)\le\gamma_1,\ldots,g_m(x)\le\gamma_m, \ \text{ and } h_1(x)=\kappa_1,\ldots,h_l(x)=\kappa_l. \end{array} \tag{1}$$

Here f is the objective function and the $g_i(x)\le \gamma_i$ respectively $h_j(x) = \kappa_j$ are the $(m+l)$ restrictions. The g_i and the h_j are called restriction functions. The values γ_i and κ_j are called capacities. The components of the vector x are called decision variables. The set of feasible vectors x are denoted by X. So, we can formalize our problem to:

$$\begin{array}{rl} \text{Find a specific vector} & \bar{x}=(\bar{x}^1,\ldots,\bar{x}^n)^T \in X \\ \text{such that} & f(\bar{x}) \ge f(x) \text{ for all } x\in X \end{array} \tag{2}$$

with $X=\{x \mid g_1(x)\le\gamma_1, \ldots, g_m(x)\le\gamma_m \text{ and } h_1(x)=\kappa_1, \ldots, h_l(x)=\kappa_l\}$.

Depending on the mathematical features of those functions, these problems are classified into certain categories.

What we have stated in (2) is a typical general Nonlinear Programming Problem (see *Nonlinear Programming*).

This survey is dedicated to a special form of that problem, namely, to Linear Programming Problems or Linear Programs (LP).

Their general formulation is:

$$\begin{array}{lll} \text{maximize} & c^T x & \\ \text{subject to} & a_1^T x \le b^1, \ldots, a_m^T x \le b^m & \text{resp. } Ax \le b, \\ \text{and} & d_1^T x = p^1, \ldots, d_k^T x = p^k & \text{resp. } Dx = p, \\ \text{where} & x, c, a_1, \ldots, a_m, d_1, \ldots, d_k \in \mathbb{R}^n,\ b \in \mathbb{R}^m,\ p \in \mathbb{R}^k. & \end{array} \tag{3}$$

Here the vectors a_i^T are the row vectors of the matrix $A \in \mathbb{R}^{(m,n)}$ and the vectors d_i^T form the matrix D.

Instead of the general functions $f(x)$, $g_i(x)$ and $h_j(x)$, we now have linear functions $c^T x$, $a_i^T x$ and $d_j^T x$. The functions c, a_i, d_j are the gradients of f, g_i, h_j.

An important property of these problems lies in the fact that the variables may attain all real values and that they may vary continuously. Furthermore, the contribution of one variable to the objective function or to the restriction functions is proportional to its value in this specific feature of linearity.

This is different in another type of problem, which will repeatedly be mentioned in this text as a closely related, but discretely structured type, namely the Integer Linear Optimization Problem (see *Combinatorial Optimization and Integer Programming*), which is:

$$\begin{array}{lll} \text{maximize} & c^T x & \\ \text{subject to} & a_1^T x \le b^1, \ldots, a_m^T x \le b^m & \text{resp. } Ax \le b, \\ \text{and} & d_1^T x = p^1, \ldots, d_k^T x = p^k & \text{resp. } Dx = p, \\ \text{and} & x \in \mathbb{Z}^n & \text{(all } x^i \text{ are integers)} \\ \text{where} & x, c, a_1, \ldots, a_m, d_1, \ldots, d_k \in \mathbb{R}^n,\ b \in \mathbb{R}^m,\ p \in \mathbb{R}^k. & \end{array} \tag{4}$$

1.2. Examples

The variety of problems of the LP-type is overwhelming. Here we can only mention a few of the different applications, where linear optimization problems must be solved.

1. **Ingredient-Mixing**

   Assume that a product is a collection of different ingredients $I_1, \ldots, I_n$. Then assume that the cost of each ingredient is different, and that the quality can be controlled by varying the weights of the n ingredients. A certain level of quality shall be assured. This can be expressed by certain restrictions about the feasible mixtures. Then, it is our aim to minimize the costs by choosing the feasible mixture that has cheapest cost.

2. **Profit-Maximization in Investment**

   Assume that certain investment facilities are available for a given amount of money. How shall we distribute our money in order to maximize our profit, when the investments are bounded, a certain level of risk shall not be exceeded, and a certain diversification is required?

3. **Maximizing the Delivered Amount**
   Assume that several pipelines or cables are available to send goods, liquids, messages, or data from a sender to a receiver continuously. Which pipelines or cables, and to what extent shall they be used in order to maximize the amount that arrives at the receiver's side per time unit?

4. **Production Planning**

   Assume that the management of a company has to decide what quantities of certain products shall be manufactured. It is desirable to maximize the resulting profit without exceeding the capacity of available machine-time, labor force, financial credit, etc

5. **Transportation**
   Assume that m stores with certain stockpiles have to deliver goods to k shops, which have specific demands. Further, suppose that a delivery from store i to shop j incurs costs of c_{ij} per unit. How shall the transport be organized? How much of the goods shall be delivered from store i to shop j in order to minimize the costs and to satisfy the demands of all the shops?

These are typical linear optimization problems. However, linear optimization techniques not only help solve these pure applications of Linear Programming, they also serve as extremely helpful tools. They can be used as subroutines in the calculation of Integer Programming Problems, as for example: staff scheduling of airline flights, flight or travel scheduling, route-planning, packing problems, location problems, etc. In such cases, one uses Linear Programming repeatedly as a subroutine on subproblems where the integrity-condition is ignored. A systematic exploitation of the insights achieved in that way leads to the optimum integer point (see *Combinatorial Optimization and Integer Programming*, *Scheduling Problems, Routing Problems, Graph and Network Optimization*).

Following is a typical numerically specified example problem:

$$
\begin{array}{lrcrcrcr}
\text{maximize} & & & 3x^2 & & & & \\
\text{subject to} & -1.03x^1 & + & 0.12x^2 & + & 0.06x^3 & \leq & 0.32 \\
 & 0.05x^1 & - & 1.06x^2 & + & 0.06x^3 & \leq & 0.52 \\
 & 0.011x^1 & + & 0.03x^2 & - & 1.09x^3 & \leq & 0.47 \\
 & 1.14x^1 & + & 0.15x^2 & + & 0.11x^3 & \leq & 0.722 \\
 & 0.2x^1 & + & 1.14x^2 & + & 0.075x^3 & \leq & 0.672 \\
 & 1.14x^1 & + & 0.17x^2 & + & 1.01x^3 & \leq & 0.532 \\
 & -0.22x^1 & - & 0.56x^2 & + & 0.67x^3 & \leq & 0.25 \\
 & 1.1x^1 & - & 0.3x^2 & - & 1.31x^3 & \leq & 0.80 \\
 & -1.1x^1 & + & 0.9x^2 & + & 0.9x^3 & \leq & 0.85
\end{array} \tag{5}
$$

This problem can equivalently be written as:

maximize $c^T x$ subject to $Ax \leq b$, where

$$
c = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \quad \text{and } A = \begin{pmatrix} -1.03 & 0.12 & 0.06 \\ 0.05 & -1.06 & 0.06 \\ 0.011 & 0.03 & -1.09 \\ 1.14 & 0.15 & 0.11 \\ 0.2 & 1.14 & 0.075 \\ 0.14 & 0.17 & 1.01 \\ -0.22 & -0.56 & 0.67 \\ 1.1 & -0.3 & -1.31 \\ -1.1 & 0.9 & 0.90 \end{pmatrix} \quad \text{and } b = \begin{pmatrix} 0.32 \\ 0.52 \\ 0.47 \\ 0.722 \\ 0.672 \\ 0.532 \\ 0.25 \\ 0.80 \\ 0.85 \end{pmatrix} \tag{6}
$$

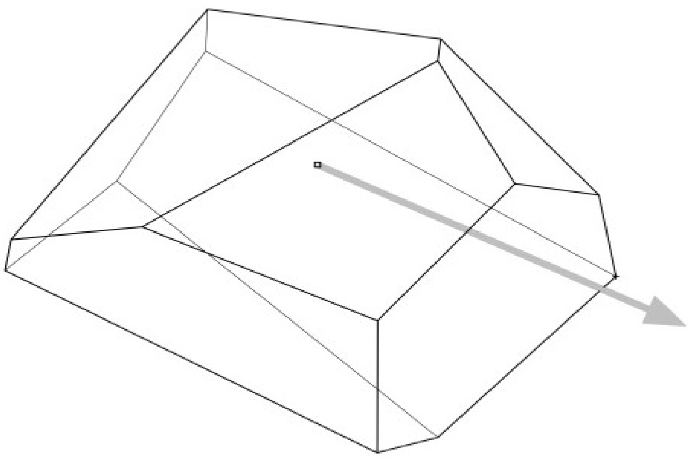

Figure 1: Feasible Region

Figure 1 shows the feasible region for the example given above. The arrow shows the direction of the second component x^2, our objective direction. The optimal point is the rightmost vertex = $(-0.258, 0.662, -0.416)^T$ and the optimal value is 1.986.

Geometrically, such a Linear Optimization Problem can be seen as follows:

Find a point $\left(x^1,\dots,x^n\right)^T \in \mathbb{R}^n$, which maximizes the scalar product c^Tx on the feasible region (polyhedron) $X = \{x \mid Ax \le b\}$. This means that among the isoclines of c^Tx, one shall be selected, which touches X without intersecting its interior. The "touching point" is the optimal point for which we have been searching.

From this view, it becomes clear that there are four qualitatively different outcomes of an LP:

1. The feasible region X is bounded and the linear (continuous) function c^Tx attains its optimal value on X.
2. The feasible region X is unbounded, but the linear function c^Tx attains its optimal value on X anyway.
3. The feasible region X is unbounded, and the linear function c^Tx has no optimal value on X, because it is unbounded from above on X itself.
4. The constraints of $Ax \le b$ are contradictory, which induces that $X = \emptyset$.

In algorithms this listing leads to the following general advice:

- First, check whether X has feasible points. If not, then STOP because of INFEASIBILITY.
- Else, try to optimize. As soon as it is clear that the objective function is unbounded, then STOP also, but this time because of UNBOUNDEDNESS.
- Else, proceed with the optimization process until an optimal point is found. Then STOP because of OPTIMALITY.

1.3. Different Forms of Programs and Transformations

So far, we have presented only linear programs of the type:

maximize c^Tx subject to $Ax \le b$.

However, all analytical and arithmetical insights about Linear Programs can easily be transferred to the "General Linear Programming Problem":

$$\begin{array}{lrcrcrcl}
\text{maximize} & d^Tx & + & e^Ty & + & f^Tz & & \\
\text{subject to} & Ax & + & By & + & Cz & \le & a \\
 & Dx & + & Fy & + & Gz & = & b \\
 & Hx & + & Iy & + & Jz & \ge & c \\
 & x & & & & & \ge & 0 \\
 & & & & & z & \le & 0
\end{array} \tag{7}$$

From the essence of a real problem, it does not matter in which structural way some restrictions have been modeled. So, we allow the following transformations and we regard their outcome as equivalent:

1. $a^T x = \beta \Longleftrightarrow a^T x \le \beta$ AND $a^T x \ge \beta$
   An equation can be replaced by two inequalities.
2. $a^T x \ge \beta \Longleftrightarrow -a^T x \le -\beta$
   An inequality can be replaced by the reverse negative inequality.
3. $a^T x \le \beta \Longleftrightarrow a^T x + y = \beta, y \ge 0$
   An inequality can be replaced by an equation with a nonnegative slack variable.
4. $x = x_+ - x_-$ *with* $x_+^i = Max\{x^i, 0\}$ *and* $x_-^i = Max\{-x^i, 0\}$
   A variable can be partitioned as a difference of two positive (minimal) parts.
5. $A_{\bullet i} x^i$ *with* $x^i \ge 0$ *and* $-A_{\bullet i} z^i$ *with* $z^i \le 0$
   The contribution of a column and a variable can be represented by the negative column and the negative variable.

Of course, one can replace a maximization problem by a minimization problem of the negative objective function (if we keep in mind that our result is negative then).

For the treatment of the different programs, it is very important to have the following reduction opportunity: In every equivalence class, according to the above mentioned transformations, there is a problem as:

$$\text{maximize} \quad c^T x \quad \text{subject to} \quad Ax \le b \quad \text{in canonical form,} \tag{8}$$

and as:

$$\text{maximize} \quad c^T x \quad \text{subject to} \quad Ax = b,\ x \ge 0 \quad \text{in standard form.} \tag{9}$$

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Biographical Sketch

Karl Heinz Borgwardt (born 1949) is Professor for Optimization and Operations Research at the Institute for Mathematics of the University of Augsburg. He has been there since 1984. He received his diploma at the University of Saarbruecken, his doctoral degree and his Habiltation at the University of

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