OPTIMIZATION IN INFINITE DIMENSIONS

Martin Brokate
Technische Universität München, Germany.

Keywords: Convex optimization, Duality, Lagrange function, Necessary optimality conditions, Optimal control, Partial differential equations, Dynamic programming, Calculus of variations, Variational method, Finite element method, Nonsmooth optimization, Optimal shape design, Shape optimization

Contents

1. Introduction
2. Infinite-Dimensional Optimization Problems
3. Convex Problems and Duality
4. Necessary Optimality Conditions
5. Optimal Control Problems
6. Calculus of Variations
7. Nonsmooth Problems
8. Optimal Shape Design
Glossary
Bibliography
Biographical Sketch

Summary

This paper gives an overview over basic mathematical settings used to tackle problems with an infinite number of free variables or constraints.

1. Introduction

Infinite-dimensional optimization problems are optimization problems where, in order to reach an optimal solution, one may either associate values to an infinite number of variables, or one has to take into account an infinite number of constraints, or both. Such problems occur naturally if the system to be optimized includes dependencies which vary continuously in time or space. Thus, all problems where one wants to minimize an energy functional which depends on continuously varying variables belong to this class. Moreover, all optimization problems which involve differential or integral equations are infinite-dimensional, at least at the outset. Even if one is really only interested in the behavior with respect to a finite number of variables, the infinite-dimensional character of the underlying system often has implications whose understanding is crucial for the solution of the problem.

Infinite-dimensional optimization problems first appeared several hundred years ago in the calculus of variations, when the differential calculus was extended to find not only several variables, but whole curves which are minimal with respect to some predefined goal. A famous example is the so-called *brachistochrone problem* first formulated by Galileo, where one wants to find a curve connecting two given points *A* and *B*, such that a frictionless mass moving from A to B under the influence of gravity reaches B in minimal time. Soon it was also discovered that the laws of nature often can be expressed in form of an optimization problem.

With the advent of the area of industrialization, man-made systems appeared in large numbers and complex forms, and optimization became urgent for economic reasons. However, since optimization problems (in particular, practical ones) on average have a more complicated structure than a system of equations, they are less accessible to solution by pencil and paper. Therefore, the invention of computers was a crucial ingredient for the widespread development of theory and algorithms in optimization. This is true in particular for infinite-dimensional optimization, which deals with problems where already the evaluation of the cost functional for a single instance of the free variables may involve the solution of a complicated system of differential equations. Indeed the progress of mathematics and computer science in the construction of hardware, algorithms and software together led to the gain of many orders of magnitude in the speed of solving optimization problems. For infinite-dimensional problems, this meant that not only larger instances of previously solved problems, but also completely new classes of problems became tractable. During the ’60s, the achievements to solve nonlinear finite-dimensional optimization problems could be combined with the solution techniques for ordinary differential equations, and optimal control problems for ordinary differential equations could be treated successfully in practically relevant situations such as trajectory optimization in aircraft and space vehicle flight. At that time, for instance, the solution of optimal control problems for partial differential equations was almost completely out of question, mainly because of the lack of computational and algorithmic power. This situation has changed dramatically during the last 20 years, so that the solution of optimal control problems has become a feasible option for a lot of situations described by partial differential equations. On the other hand, the solution of the dynamic programming problem in continuous time, that is, the solution of the Hamilton-Jacobi-Bellman equation, remains elusive except for simple academic problems.

A continuous progress also has taken place with respect to the areas where such algorithms are applied. Infinite-dimensional optimization nowadays not only pervades the engineering disciplines to a large extent, but has also found a firm footing in life sciences and economics. In particular, a large variety of infinite-dimensional problems has been studied in the subarea of population dynamics of species of all kind. As dynamical systems and continuum mechanics modeling enters the medical sciences, it is a safe prediction that optimization and optimal control will play an increasing role not only in the engineering of the relevant technology, but also in bioengineering explicitly focused on processes which take place within living tissue.

It is not the intention of this article to describe those applications in detail. Rather, it focuses on the basic principles of optimization as they appear in infinite-dimensional problems. The examples given are as simple as possible in order to illustrate a certain feature. In Section 2, we will present an abstract framework and exhibit some additional difficulties encountered in infinite-dimensional problems. Sections 3 and 4 are devoted to duality in convex problems and optimality conditions in nonconvex problems. We retain the general formulation but will not discuss in detail the mathematical ramifications involved in a complete formal solution. Using the language and notions of functional analysis, this can be done of course, and the interested reader should consult the monographs listed in the bibliography. In Sections 5 and 6 we present some typical basic problems of optimal control and of the calculus of variations, which form the main thrust of infinite-dimensional optimization. Section 7 is devoted to nonsmooth problems, in which we do not have any or enough derivatives available. Section 8 discusses optimal shape design problems, where the shape itself of the underlying domain of the system constitutes the free variables for optimization.

To actually solve infinite-dimensional problems on the computer, we of course have to reduce them to finite-dimensional form by discretization. The solution of the latter is not discussed in this article, we refer to the corresponding chapters and, again, to the bibliography.

-
-
-

Bibliography

Ciarlet P.G. and Lions J.-L. editors. (1990-2000). *Handbook of numerical analysis I - VII*. Amsterdam, Holland: North-Holland. [Covers a large amount of numerical methods for finite and infinite dimensional problems]

Clark C.W. (1990). *Mathematical Bioeconomics. The Optimal Management of Renewable Resources*. New York: Wiley. [Treats the maximum principle and its applications in biology, environmental science and economics]

Clarke F.H. (1983). *Optimization and Nonsmooth Analysis*. New York: Wiley. Reprinted 1989 by Les publications CRM, Université de Montréal, and 1990 by SIAM. [Seminal introduction to nonsmooth problems]

Dautray R. and Lions J.-L. (1990-1993). *Mathematical Analysis and Numerical Methods for Science and Technology*. Berlin: Springer-Verlag. [Treats analysis and numerical solution of initial and boundary problems, especially from continuum mechanics and electrodynamics]

Ekeland I. and Temam R. (1976). *Convex Analysis and Variational Problems*. Amsterdam, Holland: North Holland. [Concise presentation of convex analysis and optimization and its connections to the calculus of variations]

Fattorini H.O. (1999). *Infinite-Dimensional Optimization and Control Theory*, Encyclopedia of Mathematics and Its Applications, 62. Cambridge: Cambridge University Press. [Treats the theory of optimal control with emphasis on optimality conditions, partial differential equations and relaxed solutions]

Fleming W.H. and Rishel R.W. (1975). *Deterministic and Stochastic Optimal Control*. Berlin: Springer-Verlag. [Classical introductory textbook, includes also the stochastic maximum principle]

Haslinger J. and Neittaanmäki P. (1996). *Finite Element Approximation for Optimal Shape, Material and Topology Design*. Chichester, England: Wiley. [Discusses theory and numerical solution of optimal shape design problems]

Hiriart-Urruty J.-P. and Lemaréchal C. (1993). *Convex Analysis and Minimization Algorithms I, II*. Berlin: Springer-Verlag. [Represents a detailed and vivid introduction into all aspects of convex optimization restricted, however, to finite dimensions]

Rockafellar R.T. (1970). *Convex Analysis*. Princeton: Princeton University Press. Reprinted 1997 as paperback. [Treats the theory of convex analysis and convex optimization, again, restricted to finite dimensions]

Sokolowski J. and Zolesio J.-P. (1992). *Introduction to Shape Optimization. Shape Sensitivity Analysis*. Berlin: Springer-Verlag. [Describes the theory and the calculus of shape optimization]

Sontag E.D. (1990). *Mathematical Control Theory*. New York: Springer-Verlag. [Standard textbook on all fundamental aspects of control theory for ordinary differential equations]

Zeidler E. (1986-1990). *Nonlinear Functional Analysis and its Applications I - IV*. New York: Springer-Verlag. [Covers the theory of functional analysis of nonlinear problems as well as applications in many different areas of science.]

Biographical Sketch

Martin Brokate is Full Professor in the Faculty of Mathematics of the Technical University of Munich, Germany. He received his diploma and his doctoral degree in Mathematics from the Free University of Berlin and his habilitation in Mathematics from the University of Augsburg. He has done extensive research in mathematical analysis and optimization, in particular, in the area of optimal control theory and of mathematical modeling of hysteresis phenomena. He has published more than forty papers. He is the author of the book *Optimal Control of OrdinaryDifferential Equations with Nonlinearities of Hysteresis Type* and, together with J. Sprekels, is an author of the book *Hysteresis and Phase Transitions.*