THE PRINCIPLES OF THE CALCULUS OF VARIATIONS

Michael Grüter
Universität des Saarlandes, Saarbrücken, Germany.

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Summary

Assuming the existence of a classical solution for a variational integral one derives a system of second order differential equations, the Euler-Lagrange equations, which necessarily have to be satisfied. To ensure the existence of, for example, a minimizer one uses the direct method of the calculus of variations. This produces a generalized solution and therefore the question of regularity arises. Here, the convexity of the variational integral is an important feature. One method to tackle non-convex problems is the theory of Γ-convergence. Topological conditions enter when investigating unstable critical points.

1. Introduction

As *Giaquinta* and *Hildebrandt* write in the introduction to the first volume of their treatise: "*The Calculus of Variations is the art to find optimal solutions and to describe their essential properties*." Examples from daily life are: which object has some property to a highest or lowest degree, or what is the optimal strategy to reach some goal. The *Isoperimetric Problem*, already considered in antiquity, is one such question: *Among all possible closed curves of a given length, find those for which the area of the enclosed inner region is maximal*. A property shared by such optimum problems consists in the fact that, usually, they are easy to formulate and to understand, but much less easy to solve.

The principle of economy of means: *"What you can do, you can do simply"* is an idea that dominates many of our everyday actions as well as the most sophisticated inventions or scientific theories. Therefore, it should come as no surprise that this idea was extended to the area of natural phenomena. As *Newton* wrote in his *Principia*: "*Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased with simplicity and affects not the pomp of superfluous causes*." Similarly, in the first treatise on the Calculus of Variations, his *Methodus inveniendi* from 1744, *Euler* wrote: "*Because the shape of the whole universe is most perfect and, in fact, designed by the wisest creator, nothing in all of the world will occur in which no maximum or minimum rule is somehow shining forth*." Even in the rational world of today’s science where apparently no metaphysics is involved, there remains the fact that many if not all laws of nature can be given the form of an extremal principle.

Apart from this introduction, this article is divided into three sections. The first one, *Classical Theory*, roughly covers the time from *Euler* to the end of 19th century and is concerned with so called *Indirect Methods*. The next section describes the relevant ideas developed during the last 100 years and is entitled *Direct Methods*. An important ingredient here is the introduction of functional analytic techniques. In fact, it was the Calculus of Variations, which gave birth to the theory of Functional Analysis. The third and final, extremely short, section bears the title *Unstable Critical Points*, and is concerned with equilibrium solutions, which are no longer extrema. Here, an important role is played by topological methods. In this overview of the *Principles of the Calculus of Variations*, it was of course neither intended nor possible to cover all the important contributions to the subject. However, the list of references includes some of these. The material presented here is restricted to exemplary model cases. Thus, for example, variational integrals depending on higher derivatives or variational problems with subsidiary conditions are not included.

2. Classical Theory

Compared to the developments in the 20th century, which will be the topic of section 3, this part of the calculus of variations could also be called *"Indirect Methods"*. The underlying idea is the following: Suppose you know that a solution to a variational problem (e.g. a minimum) exists. What can you say about such a solution? Which equation(s) does it satisfy? Which properties (e.g. symmetry) of the corresponding variational functional does it inherit?

2.3 The finite dimensional case

First, let us have a look at the finite dimensional situation. Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}$ a smooth function. Suppose f has a *local minimum* at a point $x_0 \in \Omega$, i.e. there is a ball $B_r(x_0) \subset \Omega$, $r > 0$, such that

$$f(x) \geq f(x_0) \text{ for any } x \in B_r(x_0). \tag{1}$$

Then, at such a point $x_0 \in \Omega$ we have

$$Df(x_0) = grad\ f(x_0) = 0 \tag{2}$$

where grad $f(x_0) \in \mathbb{R}^n$ is the vector whose components are the partial derivatives of f at x_0.

A point $x_0 \in \Omega$ satisfying (2) is called a *critical point* of f.

Furthermore, using second derivatives, we have:

a. If x_0 is a *minimal point* of f then $D^2f(x_0) \geq 0$, i.e. the symmetric matrix of second partial derivatives is *positive semidefinite*.
b. Suppose, x_0 is a *critical point* of f and furthermore that $D^2f(x_0) > 0$ (it positive definite) then x_0 is a *minimal point* of f.

2.4 One-Dimensional Variational Integrals

Let us now turn to the *calculus of variations*. We start with one-dimensional integrals, that is we consider functionals $\mathcal{F}$ of the form

$$\mathcal{F}[u] = \int_I F(x, u(x), u'(x))dx. \tag{3}$$

Such functionals are called *variational integrals*. Here, $I \subset \mathbb{R}$ is an interval (in general I will be bounded), $F : \bar{I} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is called the *Lagrange function* (we write $F = F(x, z, p)$), and $u : \bar{I} \to \mathbb{R}^N$ is supposed to be smooth. More generally, it suffices to consider the case $F \in C^1(U)$ with $U \subset \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ an open set such that $\{(x, u(x), u'(x)) : x \in \bar{I}\} \subset U$. In this case, $\mathcal{F}[v]$ is defined for any $v \in C^1(\bar{I}, \mathbb{R}^N)$ provided $\| v - u \|_{C^1(\bar{I})} < \delta$ for $\delta > 0$ sufficiently small. Thus, for an arbitrary function $\varphi \in C^1(\bar{I}, \mathbb{R}^N)$ we see that

$$\Phi(\varepsilon) := \mathcal{F}[u + \varepsilon\varphi] \tag{4}$$

is defined as soon as $|\varepsilon| < \varepsilon_0 := \delta / \|\varphi\|_{C^1(\bar{I})}$. We get $\Phi \in C^1(-\epsilon_0, \epsilon_0)$ and easily compute

$$\Phi'(0) = \int_I \{F_z(x,u,u')\cdot\varphi + F_p(x,u,u')\cdot\varphi'\}dx. \tag{5}$$

In the following we call $\delta\mathcal{F}[u,\ \varphi] := \Phi'(0)$ the *First Variation of* $\mathcal{F}$ *at u in direction* φ. From (5) we deduce that $\delta\mathcal{F}[u,\ \varphi]$ is – with respect to φ – a *linear functional* on $C^1(\bar{I}, \mathbb{R}^N)$.

Definition

A function $u \in C^1(I,\ \mathbb{R}^N)$ satisfying

$$\int_I \{F_z(x,u,u')\cdot\varphi + F_p(x,u,u')\cdot\varphi'\}\,dx = 0 \tag{6}$$

for any $\varphi \in C_c^\infty(I,\mathbb{R}^N)$ is called a *weak* C^1*-extremal* of $\mathcal{F}$. Note, that for $u \in C^1(\bar{I}, \mathbb{R}^N)$ we have that (6) is equivalent to the fact that $\delta\mathcal{F}[u,\ \cdot] \equiv 0$ on $C_c^\infty(I,\mathbb{R}^N)$
With the above definition in mind, we have the following first model result:

Theorem 1

Suppose, that $u \in C^1(\bar{I},\ \mathbb{R}^N)$ is a *weak minimizer* of $\mathcal{F}$, that is

$$\mathcal{F}[u] \le \mathcal{F}[u+\varphi] \tag{7}$$

for any $\varphi \in C_c^\infty(I,\mathbb{R}^N)$ such that $\|\varphi\|_{C^1(\bar{I})} \le \delta$ for some $\delta \in (0,1)$.

Then, u is a weak C^1 – extremal of $\mathcal{F}$.

For the following considerations we assume that u and F are at least of class C^2. A partial integration in (6) then implies ($I=(a,\ b)$)

$$0 = \int_a^b \{F_z(x,u(x),u'(x)) - \frac{d}{dx}[F_p(x,u(x),u'(x))]\}\cdot\varphi(x)dx \tag{8}$$

for any $\varphi \in C_c^\infty(I,\mathbb{R}^N)$.

We now need the so called

Fundamental Lemma (of the Calculus of Variations)

If $h \in C^0(I,\mathbb{R}^N)$ such that for any $\varphi \in C_c^\infty(I,\mathbb{R}^N)$ we have

$$\int_a^b h(x)\cdot\varphi(x)dx = 0, \tag{9}$$

then $h \equiv 0$ on $I = (a, b)$.

Because of the importance of this result in the calculus of variations, we present the simple proof below.

Proof

We argue by contradiction, that is we assume there exists $i_0 \in \{1, ..., N\}$ and $x_0 \in (a, b)$ such that $h^{i_0}(x_0) \neq 0$. The *continuity* of h then yields the existence of some number $\delta > 0$ with $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ such that

$$|h^{i_0}(x)| > \frac{1}{2}|h^{i_0}(x_0)| \quad \text{for } |x - x_0| < \delta. \tag{10}$$

Now, choose $\eta \in C_c^\infty(I, \mathbb{R}^N)$ in such a way that

$$\begin{aligned} &\eta^{i_0}(x) \equiv 0 \text{ for } |x - x_0| \geq \delta, \\ &\eta^{i_0}(x) > 0 \text{ for } |x - x_0| < \delta, \\ &\eta^i(x) \equiv 0 \text{ for } i \neq i_0. \end{aligned} \tag{11}$$

Finally, define φ by $\varphi(x) := h^{i_0}(x_0)\eta(x)$ so that from (9) we get]

$$\begin{aligned} 0 = \int_a^b h(x)\cdot\varphi(x)dx &= \int_{x_0-\delta}^{x_0+\delta} h(x)h^{i_0}(x_0)\eta(x)dx > \\ &> \frac{1}{2}|h^{i_0}(x_0)|^2 \int_{x_0-\delta}^{x_0+\delta} \eta^{i_0}(x)dx > 0. \end{aligned} \tag{12}$$

This is a *contradiction* and concludes the proof.

Remark

An important *generalization* of the fundamental lemma reads as follows: *Suppose* $h \in L^1(I, \mathbb{R}^N)$ (*instead of* $C^0(I, \mathbb{R}^N)$) *satisfies* (9)*, then* $h(x) = 0$ *for* $\mathcal{L}^1$ –a.e. $x \in I$. The proof is similar and uses the fact that $C_c^\infty(I, \mathbb{R}^N)$ is dense in $L^2(I, \mathbb{R}^N)$.

Because of the fundamental lemma we get

Theorem 2

Suppose $u \in C^2(I, \mathbb{R}^N)$ is a weak extremal of $\mathcal{F}$ and that F is of class $C^2(U)$ where U is an open set containing the 1–graph of u. We then have

$$\frac{d}{dx}[F_p(x,u(x),u'(x))] - F_z(x,u(x),u'(x)) = 0 \text{ on } I. \tag{13}$$

Remark

Note, that (13) is a *system of ordinary differential equations*, the so called *Euler-Lagrange equations:*

$$\frac{d}{dx}[F_{p^i}(x,u(x),u'(x))] - F_{z^i}(x,u(x),u'(x)) = 0, \quad i = 1, \ldots, N. \tag{14}$$

To be more precise, we get a system of N quasilinear ordinary differential equations of second order for the N unknown functions $u^1, \ldots, u^N$.

At this point, let us discuss several examples.

Examples

1. The Lagrange function $F(x, z, p) = \omega(x, z)\sqrt{1+|p|^2}$ with $N = 1$ and $\omega > 0$ leads to the variational integral

$$\mathcal{F}[u] = \int_a^b \omega(x,u)\sqrt{1+(u')^2}\,dx \tag{15}$$

and the Euler–Lagrange equation

$$\frac{d}{dx}\left[\omega(x,u)\frac{u'}{\sqrt{1+(u')^2}}\right] - \omega_z(x,u)\sqrt{1+(u')^2} = 0. \tag{16}$$

This can be written as

$$\kappa\omega\sqrt{1+(u')^2} = \omega_z - u'\,\omega_x \tag{17}$$

where

$$\kappa := \frac{d}{dx}\left[\frac{u'}{\sqrt{1+(u')^2}}\right] \tag{18}$$

is the *curvature* of the curve graph $u \subset \mathbb{R}^2$.

In case $\omega \equiv 1$ the variational integral $\mathcal{F}$ is just the *length* of graph u and we get $\kappa \equiv 0$, i.e. $u'' \equiv 0$. Thus, the weak extremals of class C^2 of the length functional are the (affine) linear functions $u(x) = \alpha x + \beta$ ($\alpha, \beta \in \mathbb{R}$).

2. The choice $F(x, z, p) = F(p) = \frac{1}{2} | p |^2 = \frac{1}{2} \sum_{i=1}^{N} | p^i |^2$, $N \geq 1$, leads to *Dirichlet's integral*:

$$\mathcal{D}[u] = \frac{1}{2} \int_a^b | u' |^2 \, dx \tag{19}$$

and the Euler–Lagrange equations

$$\frac{d}{dx}(u^i)' = (u^i)'' = 0, i = 1, \ldots, N. \tag{20}$$

Again, we identify the extremals as the affine linear functions.

3. A *classical problem* in the calculus of variations is the so called *brachistochrone problem* first formulated by Galileo in 1638:

Find a curve, connecting two given points A and B, on which a point mass moves without friction under the influence of gravity in the least possible time from the initial point A to the end point B below A.

Galileo believed the optimal curve to be a circular arc. However, this is wrong and the correct solution was finally found by Johann Bernoulli in 1697:

Suppose, that in a Cartesian coordinate system with gravity acting in direction of the negative y-axis, $A = (x_1, y_1)$, $B = (x_2, y_2)$, $x_1 < x_2$, $y_1 > y_2$. Then, for a function $u : [x_1, x_2] \to \mathbb{R}$ with $u(x_i) = y_i$ and $u(x) < y_1$ for $x \in (x_1, x_2]$, the time needed by the point mass to slide from A to B along the graph of u, starting at A with zero velocity, is given by the quantity

$$\mathcal{F}[u] = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + | u'(x) |^2}{y_1 - u(x)}} \, dx \tag{21}$$

where g denotes the acceleration due to gravity.

The *solution* turns out to be a *cycloid*, which in parametric form can be given as

$$\begin{cases} x(t) = x_1 + k(t - \sin\, t), \\ u(t) = y_1 - k(1 - \cos\, t). \end{cases} t \in [0, T] \tag{22}$$

Here, the constants k and T are determined by the conditions $x(T) = x_2$ and $u(T) = y_2$.

In addition to the Euler–Lagrange equations there are further conditions for a minimum (compare the beginning of this chapter).

The *necessary Legendre condition*:

$$\sum_{i,k=1}^{N} F_{p^i p^k}(x,u(x),u'(x))\xi^i\xi^k \geq 0 \qquad (23)$$

for any vector $\xi \in \mathbb{R}^N$ and every $x \in \overline{I}$. (Follows from $\Phi''(0) \geq 0$.)

The *sufficient Legendre-condition*:

There is a number $m > 0$ such that

$$\sum_{i,k=1}^{N} F_{p^i p^k}(x,z,p)\xi^i\xi^k \geq m\,|\,\xi\,|^2 \qquad (24)$$

for any vector $\xi \in \mathbb{R}^N$ and every (x, z, p) $\in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$.

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Biographical Sketch

Michael Grüter is Professor of Mathematics at the University of the Saarland at Saarbrücken in Germany. He received his Diplom degree in Mathematics and Philosophy from the University of Bonn. He received his doctoral degree and Habilitation in Mathematics from the University of Düsseldorf. He has done extensive research in the areas of partial differential equations, the calculus of variations, and geometric measure theory. His main interests are in the geometric calculus of variations, in particular in minimal surfaces and harmonic mappings. Among Dr. Grüter's publications are articles in *Acta Mathematica*, *Annali della Scuola Normale Superiore di Pisa*, and *Journal für die Reine und Angewandte Mathematik*. Until 2001 he has been on the editorial boards of the textbook series *Aufbaukurs Mathematik* and the series *Advanced Lectures in Mathematics,* both published by *Vieweg Verlag*. Dr. Grüter was one of the organizers of the *First European Conference on Elliptic and Parabolic Problems* held in 1991. He has been a member of the *Sonderforschungsbereich 256 Nonlinear Partial Differential Equations* in Bonn.