INVESTMENT MODELS

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Contents

Summary

In this paper we exposit investment models in discrete and continuous time with a special emphasis on solution methods. In a first part, we describe the early work of Markowitz and Tobin. We define and solve the mean-variance portfolio selection problem and formulate the two-fund separation theorem. Multiperiod discrete-time models are investigated by stochastic dynamic programming. For continuous-time portfolio problems, we explain two solution methods: stochastic control techniques and the martingale approach. In particular, we derive optimal portfolios for special HARA-utilities. The last section identifies some extensions and further models.

1. Introduction

The heart of investment models is the selection of an optimal set of financial assets. An investor endowed with a given amount of income has to decide how many shares of which asset he should hold at which time instant in order to maximize his wealth at the time horizon T and/or his expected utility of consumption during the time interval $[0, T]$. The main objective of investment analysis is to provide optimal decisions for such portfolio selection problems.

The earliest approach for solving the portfolio selection problem is the so-called mean-variance formulation. It was pioneered by Markowitz (1952, 1959) and Tobin

(1958) and is only suited for one-period problems. It still has great importance in real-life applications and is widely applied in risk management departments of banks. The problem of multiperiod portfolio selection was proposed in the late sixties and early seventies by (among others) Mossin (1968), Samuelson (1969) and Fama (1970) in a discrete-time setting. The work of Merton (1969, 1990) must be regarded as the real starting point of continuous-time portfolio theory.
In what follows we will distinguish investment models in discrete and continuous time. In section 2, we describe the mean-variance approach. Section 3 is devoted to discrete-time multiperiod portfolio selection, section 4 deals with solution methods for continuous-time models.

2. Mean-Variance Portfolio Selection

In the formulation of the mean-variance portfolio, we use the following notation: x is a vector whose components denote the weight or proportion of the investor's wealth allocated to the n assets in the portfolio. Obviously, the sum of these weights is equal to 1. e is a vector of ones. r is the vector of expected returns of the n assets, where it is assumed that not all elements of r are equal, and Q is the $n \times n$ covariance matrix. We assume that Q is nonsingular. This essentially requires that none of the asset returns be perfectly correlated with the returns of any portfolio made up of the remaining assets; and that none of the assets or portfolios of the assets be riskless. Note that Q is symmetric and positive definite being a covariance matrix. Finally, r_p is the investor's target return.

2.3 Markowitz Model

Following Markowitz (1952) and Tobin (1958) the portfolio selection problem can be stated as:

$$\begin{aligned} &\text{minimize} && x^T Qx \\ &\text{subject to} && x^T r = r_p \\ & && x^T e = 1 \end{aligned} \tag{1}$$

In problem (1), we minimize the portfolio variance subject to two constraints: first, the portfolio mean must be equal to the target return r_p, and second, the portfolio weights must sum to unity, which means that all wealth is invested. Technically, we minimize a quadratic function subject to linear constraints. Since x^TQx is strictly convex, the problem (1) has a unique solution and we only need to consider the first-order conditions (see *Nonlinear Programming*).

Consider the Lagrange function of (1)

$$L = x^T Qx - \lambda_1 (x^T r - r_p) - \lambda_2 (x^T e - 1). \tag{2}$$

The first-order conditions are

$$\frac{\partial L}{\partial x} = 2Qx - \lambda_1 r - \lambda_2 e = 0 \tag{3}$$

$$\frac{\partial L}{\partial \lambda_1} = r_p - x^T r = 0 \tag{4}$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - x^T e = 0 \tag{5}$$

Solving for $x \in \mathbb{R}^n$ we obtain

$$\begin{aligned} x &= \frac{1}{2} Q^{-1}(\lambda_1 r + \lambda_2 e) = \frac{1}{2} Q^{-1}[r\ e][\lambda_1\ \lambda_2]^T \\ &= Q^{-1}[r\ e]A^{-1}[r_p\ 1]^T \end{aligned} \tag{6}$$

Where

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} r^T Q^{-1} r & r^T Q^{-1} e \\ r^T Q^{-1} e & e^T Q^{-1} e \end{pmatrix} \tag{7}$$

Then x is called a *minimum variance portfolio*. Note that x is linear in its expected return r_p. The result of this analysis can be stated as:

Theorem 2.1

Let Q be the positive definite covariance matrix, and r the vector of expected returns of the n assets where it is assumed that not all elements of r are equal. Then the minimum variance portfolio with given target return r_p is unique and its weights are given by (6).

Substituting (6) into the definition of the portfolio variance yields

$$\begin{aligned} \sigma_p^2 &:= x^T Qx = [r_p\ 1]A^{-1}[r_{p\,1}]^T \\ &= \frac{a - 2br_p + cr_p^2}{ac - b^2}. \end{aligned} \tag{8}$$

In (8) the relation between σ_p^2 and any given expected return r_p is expressed as a parabola and is called the *minimum variance portfolio frontier*. The set of portfolios having the highest return for a given variance is called set of mean-variance *efficient portfolios*. The *global minimum variance portfolio* is the portfolio with the smallest possible variance for any expected return. Its expected return r_g ist given by $r_g = \frac{b}{c}$ and its variance σ_g^2 is equal to $\frac{1}{c}$. The weights of the global minimum variance portfolio are

$$x_g = \frac{Q^{-1}e}{c}. \tag{9}$$

Equation (6) shows a *two-fund separation theorem*, that the minimum variance portfolio frontier can be generated by any two distinct frontier portfolios.

Theorem 2.2 (Two-fund separation)

Let x_a and x_b be two minimum variance portfolios with expected return r_a and r_b respectively, such that $r_a \neq r_b$.

a. Then every minimum variance portfolio is a linear combination of x_a and x_b.
b. Conversely, every portfolio, which is a linear combination of x_a and x_b, is a minimum variance portfolio.
c. In particular, if x_a and x_b are minimum variance efficient portfolios, then $\alpha x_a + (1 - \alpha)x_b$ is also a minimum variance efficient portfolio for $\alpha \in [0, 1]$.

Only two portfolios are sufficient to describe the entire efficient set. It is of practical interest to select two portfolios whose means and variances are easy to compute. One such portfolio is the global minimum variance portfolio x_g. The other efficient portfolio could be

$$x_m = \frac{Q^{-1}r}{b} \tag{10}$$

with expected return $r_m = \frac{a}{b}$ and variance $\sigma_m^2 = \frac{a}{b^2}$.

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Bibliography

Ingersoll J.E. (1987). *Theory of Financial Decision Making*, 474 pp. Totowa, NJ: Rowman and Littlefield. [Textbook on multiperiod investment models.]

Korn R. (1997). *Optimal Portfolios: Stochastic Models for Optimal Investment and Risk Management In Continuous Time*,338 pp . Singapore: World Scientific. [Recent book on continuous-time portfolio theory.]

Luenberger D.G. (1998). *Investment Science*, 494 pp. New York: Oxford University Press. [Excellent textbook on investment models.]

Markowitz H. (1959). *Portfolio Selection: Efficient Diversification of Investments*, 344 pp. New York: Wiley. [Early textbook on portfolio selection.]

Merton R.C. (1990). *Continuous-Time Finance*, 700 pp. Cambridge, MA: B. Blackwell. [First book on continuous-time portfolio theory.]

Sethi S.P. (1997). *Optimal Consumption and Investment with Bankruptcy*, 428 pp. Boston: Kluwer Academic Publishers. [Recent survey on consumption and investment models.]

Biographical Sketch

Ulrich Rieder, born in 1945, received the Ph.D. degree in mathematics in 1972 (University of Hamburg) and the Habilitation in 1979 (University of Karlsruhe). Since 1980, he has been Full Professor of Mathematics and head of the Department of Optimization and Operations Research at the University of Ulm. His research interests include the analysis and control of stochastic processes with applications in telecommunication, logistics, applied probability, finance and insurance. Dr. Rieder is Editor-in-Chief of the journal *Mathematical Methods of Operations Research*.