EXPECTED UTILITY THEORY AND ALTERNATIVE APPROACHES

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Keywords: expected utility, independence axiom, Allais paradox, common ratio effect, betweenness, weighted utility, implicit expected utility, disappointment aversion, rank-dependent utility, prospect theory, dual expected utility

Contents

Summary

In this work, we present several alternative models, which weaken the independence axiom of expected utility theory. The models are separated into two classes of models, utility theories with the betweenness property and rank-dependent models. We will also present some experimental research, which shows that also both alternative classes have a rather poor empirical performance. Therefore, theoretical as well as empirical research on decision making under risk should be continued in order to identify descriptively models that are more accurate.

1. Introduction

Since its axiomatization by John von Neumann and Oskar Morgenstern in 1947, the expected utility model has been the dominant framework for analyzing decision problems under risk and uncertainty. According to Mark Machina, this is due to "the simplicity and normative appeal of its axioms, the familiarity of the notions it employs (utility functions and mathematical expectation), the elegance of its characterizations of various types of behavior in terms of properties of the utility function (risk aversion by concavity, the degree of risk aversion by the Arrow-Pratt measure, etc.), and the large number of results it has produced."

Since the well-known paradox of Maurice Allais, however, a large body of experimental evidence has been gathered which indicates that individuals tend to violate the assumptions underlying the expected utility model systematically. This empirical evidence has motivated researchers to develop alternative theories of choice under risk and uncertainty able to accommodate the observed patterns of behavior. These models, usually termed "non-expected utility" or "generalizations of expected utility", are reviewed in this work. Because of the large number of models that appeared in the literature, we will only analyze those models, which have been designed for choice situations under risk. Another reason for this restriction is the fact that most of the important generalizations of subjective expected utility have an analogous counterpart for choice under risk.

We proceed as follows: First, the general framework and some basic definitions are introduced and then the axioms and the functional representation of expected utility are presented. We also sketch out the empirical evidence concerning the independence axiom of expected utility in order to explain the motivation for further developments.

The bulk of this work is devoted to generalizations of expected utility. Note that in accordance with the recent literature this exposition focuses mainly on weakenings of the independence axiom. Models that weaken the ordering or continuity axiom of expected utility are only briefly mentioned.

2. The General Framework

Utility theory under risk has three basic concepts: consequence, probability, and preference. Throughout this work, we will assume that the set of consequences X is given by the compact interval $[A, B] \subset \mathbb{R}^1$ since some theories are designed only for this case. Elements of X are usually interpreted as monetary amounts. The set of all probability measures over X defined on the Borel algebra $\mathcal{B}$ of X will be denoted by P. A probability measure p is a real-valued function which maps subsets of X into the interval [0, 1] and satisfies the following axioms:

$$0 \leq p(W) \leq 1 \quad \forall W \in \mathcal{B} \tag{1}$$

$$p(X) = 1 \tag{2}$$

$$p(\cup_i W_i) = \sum p(W_i) \quad \forall W_i \in \mathcal{B} \text{ which are pairwise disjoint.} \tag{3}$$

Probability measures are henceforward referred to as lotteries. In all theories considered in this work, the choice set is given by P or subsets of P. In this procedure, it is implicitly assumed that the reduction of compound lotteries axiom is satisfied because a multi-stage lottery and its reduced one-stage form define the same probability measure. Note that the set P is closed under convex mixture operations, i.e. $\lambda p + (1-\lambda)q \in P \; \forall \; \lambda \in [0, 1] \; \forall \; p, q \in P$. Thus, P satisfies the conditions of a mixture set and the mixed lottery $\lambda p + (1-\lambda)q$ assigns the probability $\lambda p(W) + (1-\lambda)q(W)$ to all $W \in \mathcal{B}$.

In some cases probability measures will be represented by their cumulative distribution functions. The set of all cumulative distribution functions over X is denoted by $D(X)$. For all $F \in D(X)$ we have $F(x) = 0$ if $x < A$ and $F(x) = 1$ if $x \geq B$. Note that there is a one-to-one correspondence between the sets P and $D(X)$, i.e. a probability measure defines a unique distribution function and vice versa.

The set of all probability measures with finite support is denoted by P^s while $\Delta \subset P^s$ consists of all degenerate probability measures, i.e. $p \in \Delta \leftrightarrow \exists \; x \in X$ with $p(x) = 1$. Elements of Δ are denoted by δ_x.

The preference of a decision maker will be formalized by the binary relation $\succsim \; \subset P \times P$. For $p, q \in P$, $p \succsim q$ indicates that p is at least as good as q (weak preference). The indifference relation ~ and the strict preference relation $\succ$ are defined from $\succsim$ by:

$$p \succ q \leftrightarrow p \succsim q \wedge \neg(q \succsim p) \qquad (4)$$

$$p \sim q \leftrightarrow p \succsim q \wedge q \succsim p, \qquad (5)$$

where, as usual, $\neg$ means "not" and $\wedge$ means "and". A binary relation $\succsim$ which satisfies:

(i) completeness: $p \succsim q \vee q \succsim p \; \forall \; p, q \in P$, and
(ii) transitivity: $(p \succsim q \wedge q \succsim r) \rightarrow p \succsim r \; \forall \; p, q, r \in P$

is defined to be an ordering. If $\succsim$ is an ordering, ~ is an equivalence relation and $\succ$ a strict partial ordering. Finally, a real-valued function $V(\cdot)$ on P is called utility function or, in mathematical terms, order homomorphism if it represents $\succsim$ on P, i.e.:

$$p \succsim q \leftrightarrow V(p) \geq V(q) \; \forall p, q \in P. \qquad (6)$$

3. Expected Utility Theory

3.3 The Theoretical Basis of Expected Utility

As mentioned in the introduction, preferences have to satisfy certain assumptions (axioms) in order to be representable by expected utility. The most basic assumption is the following ordering axiom:

<u>Ordering (O):</u>

$\succsim$ is an ordering on P, i.e. complete and transitive.

Axiom O is a fundamental tenet of rationality and is assumed in most theories of choice, even in consumer theory.

The representation of preferences by a real-valued function additionally requires a continuity assumption. We consider the following two continuity axioms:

Archimedean (AR):

$\forall\, p, q, r \in P : p \succ q \succ r \rightarrow \exists\, \lambda, \mu \in\,]0, 1[$, such that $\lambda p + (1-\lambda)r \succ q$ and $q \succ \mu p + (1-\mu)r$.

Continuity (C):

The sets $\{q \in P \mid p \succsim q\}$ and $\{q \in P \mid q \succsim p\}$ are $\forall\, p \in P$ closed in the topology of weak convergence.

Note that axiom C implies AR. Axiom AR basically rules out the possibility that one lottery is infinitely preferred to another which is necessary in order to represent preferences by a real-valued function. Without AR preferences may be lexicographic and only a vector-valued representation can be obtained. If we consider lotteries with an uncountable number of consequences AR has to be exchanged for C in order to guarantee the existence of an integral representation.

The most important axiom of expected utility theory is the independence axiom. The following variant of independence is taken from Jensen:

Independence (I):

$\forall\, p, q, r \in P : p \succ q \rightarrow \lambda p + (1-\lambda)r \succ \lambda q + (1-\lambda)r \;\forall\, \lambda \in\,]0, 1]$.

The normative appeal of the independence axiom may be best understood if $\lambda p + (1-\lambda)r$ and $\lambda q + (1-\lambda)r$ are interpreted as two-stage lotteries. Then, the probability of receiving r in the first stage is identical in both lotteries. Hence, the choice should only depend upon the preference between p and q. We are now ready to state the main result of Jensen published in 1967.

Theorem 1

Let $\succsim$ be a binary relation on P. The following statements are equivalent:

(i) $\succsim$ satisfies, O, AR, and I.
(ii) There exists a function $V : P \to \mathbb{R}$ which represents $\succsim$ on P and is linear on P, i.e.:

$$V(\lambda p + (1-\lambda)q) = \lambda V(p) + (1-\lambda)V(q) \quad \forall p, q \in P \quad \forall \lambda \in [0,1]. \tag{7}$$

Furthermore, V is unique up to positive linear transformations, i.e. another function $V^*: P \to \mathbb{R}$ represents $\succsim$ if and only if there exist real constants $a > 0$ and b such that

$$V^*(p) = aV(p) + b \quad \forall p \in P., \tag{8}$$

Property (7) of the utility function is usually labeled as linearity in the probabilities. This property is an immediate consequence of the independence axiom. The generalizations of expected utility which rest on a weakening of I do not, in general, satisfy this property. Therefore, they are non-linear utility theories.

It remains to show that theorem 1 implies that the utility of a lottery equals the expected utility of its consequences on the set P^s. Since P^s contains every degenerate probability measure, we can define a function u on X from V on P^s by

$$u(x) = V(\delta_x) \quad \forall \delta_x \in \Delta. \tag{9}$$

By a straightforward induction using property (7) we can derive:

$$V(p) = \sum_{x \in X} u(x)p(x) \quad \forall p \in P^s. \tag{10}$$

Thus, the utility of a lottery equals the expected utility of its consequences. The linearity in the probabilities can also be represented graphically by the so-called triangle diagram. If we draw attention to only three possible consequences, $x_1 \succ x_2 \succ x_3$, and define $p_2 = 1 - p_1 - p_3$ the set of all lotteries over these consequences can be represented in the (p_1, p_3)-plane. Considering a fixed utility level $\bar{V}$ and solving for p_1 yields the equation of an indifference curve:

$$p_1 \frac{u(x_2) - u(x_3)}{u(x_1) - u(x_2)} p_3 + \frac{\bar{V} - u(x_2)}{u(x_1) - u(x_2)}. \tag{11}$$

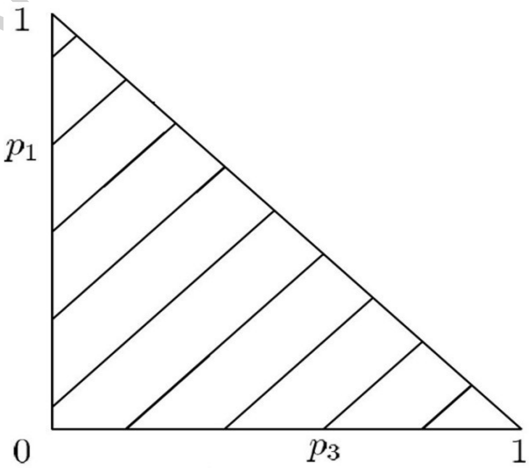

Figure 1: Expected utility

Since all the utilities are constant, (11) is a linear equation.Note that the slope is positive and independent of the utility level $\bar{V}$. Thus, indifference curves are, as depicted in Figure 1, parallel straight lines and movements in northwest direction lead to a higher utility level.

An important concept in utility theory under risk is the notion of risk aversion. In expected utility theory a preference relation is defined as displaying weak (strict) local risk aversion at $x \in X$ if $\delta_x \succsim (\succ) p$ for all non-degenerate lotteries p with an expected value of x. According to Jensen's inequality, this behavior results from a utility function which is concave on X. In order to measure the concavity of the utility function, the term $-u''(x)/u'(x)$ is often employed which is known as the Arrow-Pratt measure of absolute risk aversion. Note that the degree of risk aversion is also reflected in Figure 1, since a more concave utility function results in a higher value of $u(x_2)$ for constant $u(x_1)$ and $u(x_3)$. Thus, a higher degree of risk aversion corresponds to steeper indifference curves.

The most widely acknowledged principle of rational behavior under risk seems to be consistency of preferences with first-order stochastic dominance. A lottery F is defined to dominate a lottery G by first-order stochastic dominance ($F >_{SD} G$) if $F(x) \le G(x)\ \forall x \in X$ and $F(x) < G(x)$ for at least one $x \in X$. Consistency with stochastic dominance is demanded by the following axiom:

<u>Monotonicity (M):</u>

$p >_{SD} q \rightarrow p \succsim q$.

If the preference in axiom M is strict, we will label the axiom as strong monotonicity (SM). Preferences which can be represented by expected utility always satisfy M (SM) if the utility function is (strictly) increasing on X.

3.4 The Empirical Performance of Expected Utility

Since this work is primarily concerned with theoretical aspects in choice under risk we mention only briefly some empirical studies which observed violations of axiom I. In this section we represent lotteries by a vector $(x_1, p_1; x_2, p_2; ...; x_n, p_n)$.

Let us consider first the lotteries employed in the classical Allais Paradox where \$m denotes million \$.

$$\begin{array}{lcl} p = (\$1m, 1) & \text{versus} & q = (\$5m, 0.1; \$1m, 0.89; \$0, 0.01) \\ \hat{p} = (\$1m, 0.11; \$0, 0.89) & \text{versus} & \hat{q} = (\$5m, 0.1; \$0, 0.90) \end{array} \quad (12)$$

If we assume without loss of generality $u(\$5m) = 1$ and $u(\$0) = 0$, then $p \succ q$ implies in the expected utility framework $u(\$1m) > 0.1 + 0.89u(\$1m)$ and, therefore, $0.11u(\$1m) > 0.1$ which, in turn, implies $\hat{p} \succ \hat{q}$. In other words, expected utility theory can only accommodate the preference patterns $p \succ q$ and $\hat{p} \succ \hat{q}$ or $q \succ p$ and $\hat{q} \succ \hat{p}$. However, many analogous empirical studies have been published which report that people tend to prefer p over q and $\hat{q}$ over $\hat{p}$.

A further systematic violation of axiom I, termed common ratio effect, has been observed in experiments consisting of the following design:

$$\begin{aligned} r &= (\$x, \lambda; \$0, 1-\lambda) \qquad \text{versus} \qquad s = (\$y, \mu; \$0, 1-\mu) \\ \hat{r} &= (\$x, \gamma\lambda; \$0, 1-\gamma\lambda) \qquad \text{versus} \qquad \hat{s} = (\$y, \gamma\mu; \$0, 1-\gamma\mu) \end{aligned} \tag{13}$$

with $0 < x < y$, $\lambda > \mu$, and $\gamma \in\ ]0, 1[$. In these experiments subjects tend to state the preferences

$r \succ s$ and $\hat{s} \succ \hat{r}$, which also violate expected utility. An important special case of the common ratio effect is the certainty effect where $\lambda = 1$.

Figure 2 shows that the Allais paradox (panel A) and the common ratio effect (panel B) can be characterized in the same way since in both cases the lines joining the lottery pairs turn out to be parallel in the triangle diagram. Panel A consists of defining x_1 = \$5m, x_2 = \$1m and x_3 = \$0 and in panel B we have x_1 = \$$y$, x_2 = \$$x$ and x_3 = \$0. Additionally, Figure 2 indicates that the linear and parallel indifference curves of expected utility theory cannot accommodate the commonly observed preference patterns. Thus, in order to be compatible with the common consequence and common ratio effect, indifference curves must either get steeper in northwest direction, i.e. satisfy the fanning out hypothesis, or be non-linear.

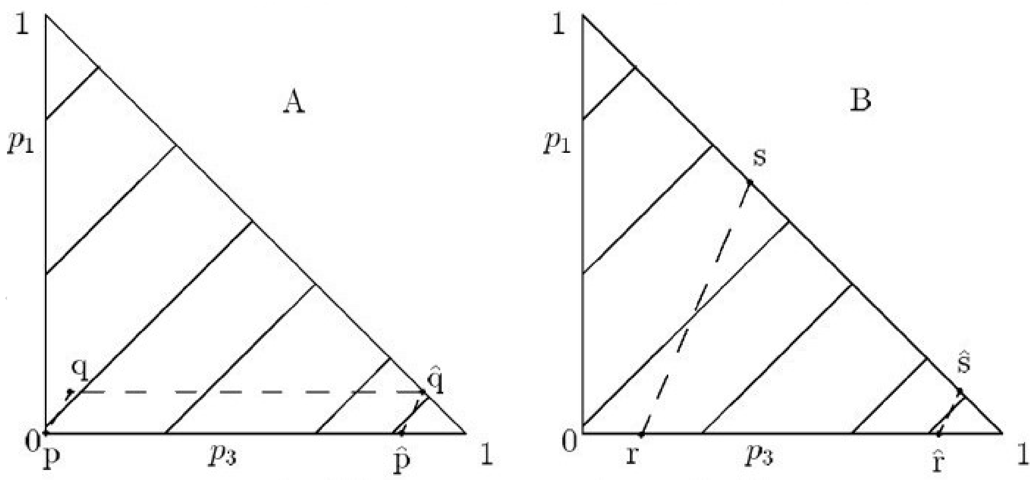

Figure 2: The common consequence and common ratio effect

The empirical observations considered in this section have been discussed thoroughly in the literature. While the results obviously indicate that expected utility does not describe actual choice behavior accurately, even the normative validity of expected utility must be questioned because many subjects did not change their preferences after arguments in favor of the independence axiom had been presented to them.

4. Non-Expected Utility Theory

4.1 Utility Theories with the Betweenness Property

4.1.1 Characterizing Betweenness

The betweenness property is implied by the independence axiom. It states that the preference for a probability mixture of two lotteries is between the preference for either lottery. Formally, betweenness is defined by the following condition:

Betweenness (BT):

$$\forall\, p, q \in P : p \sim (\succ) q \rightarrow p \sim (\succ) \lambda p + (1-\lambda) q \sim (\succ) q \;\forall\, \lambda \in \,]0, 1[.$$

If betweenness is satisfied, there is no preference for or aversion against a randomization between indifferent lotteries. For the triangle diagram, this implies that all lotteries on a line connecting two indifferent lotteries are indifferent. Thus, like independence, betweenness requires that indifference curves are straight lines or, more generally, hyperplanes. However, they are not necessarily parallel as in expected utility theory. In the case of betweenness, preferences satisfy quasiconcavity as well as quasiconvexity , which are defined as follows:

Quasiconcavity (QC):

$$\forall\, p, q \in P : p \sim q \rightarrow \lambda p + (1-\lambda) q \succsim p \;\forall\, \lambda \in [0, 1].$$

Quasiconvexity (QV):

$$\forall\, p, q \in P : p \sim q \rightarrow p \succsim \lambda p + (1-\lambda) q \;\forall\, \lambda \in [0, 1].$$

While quasiconvexity is a significant property in dynamic choice problems, quasiconcavity is a necessary assumption for the existence of a Nash equilibrium and, in conjunction with risk aversion, is a sufficient condition for a preference for portfolio diversification. Moreover, betweenness is a necessary and sufficient condition for the existence of a dominant value-revealing strategy in ascending bid auctions. In view of these results, we can conclude: Since betweenness is compatible with the primary findings of the behavioral empirical literature and retains much of the normative appeal of the independence axiom, it provides a natural candidate as an axiom for the development of alternative preference theories.

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Biographical Sketch

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