MULTIPLE-CRITERIA DECISION MAKING

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Summary

This article deals with decision making processes governed by multiple criteria. In order to concentrate on this aspect of decision making, the focus lies on deterministic approaches, at the expense of consideration of the problems of risk and uncertainty. In recent decades, an impressive number of approaches have been proposed in this field. This survey starts with a categorization of these approaches into two general classes; their main concepts are outlined and typical methods described. To assist readability, the use of mathematics is limited and the subject is developed using small typical examples of multiple-criteria decisions.

1. Introduction

Multiple-criteria decision making (MCDM) deals with decision situations where the decision maker has several—usually conflicting—objectives. In typical real-life problems, no "ideal" alternative exists in the sense of one that is optimal for each objective. Thus, the most important task in multiple criteria decision making is to find a "good" compromise. This is the alternative that performs best in the eyes of the decision maker, taking into account all objectives simultaneously. Hence, the quality of a compromise depends on the decision maker's preferences, which are multidimensional in nature and, at best, partially known. The central question in all approaches in this field is: how can additional information about the decision maker's multidimensional preference structure be used in order to support him or her in making a decision? The focus of MCDM lies in fact on decision support, rather than on a theoretical exemplification of rational choice. The following small example will be used to illustrate the basic concepts of multiple-criteria decision making.

Example: consider the problem of finding a new house for a family. The decision maker(s) might have decided upon some objectives: a lot of living space, an acceptable price, a nice residential area, and so on. Suppose they have examined the daily newspaper and compiled a list of potential houses that appear promising because they seem to meet these demands. How do they make the decision?

1.1. General Concepts

Throughout this article the focus lies on decisions. The MCDM model consists of various elements, depending on the nature of the decision problem. Figure 1 depicts the elements which are often found, and which will be described in most detail in the following subsections.

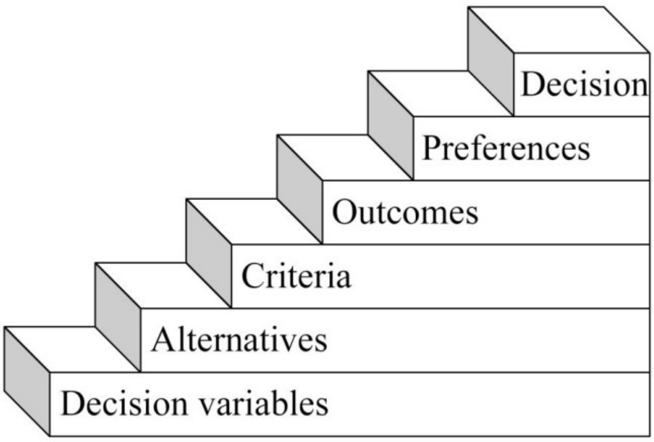

Figure 1. Elements of MCDM models

1.1.1. Decision Space, Decision Variables, and Alternatives

A decision is characterized by the decision maker's choice between different possible courses of action, called *alternatives*. In the house-purchase example the relevant courses of action are, quite simply, buying different houses. Therefore, each house identified in the newspaper forms an alternative. This yields a decision problem with a finite set of feasible alternatives, called the *feasible set* X. In this case, the decision maker is explicitly aware of each alternative. In fact, this represents a special type of decision problem, as decision problems often have an infinite number of alternatives. In this case an alternative can be defined by a vector of (real) numbers $x:=(x_1, x_2,\ldots, x_n)$. The components of this vector are called *decision variables*. Each decision variable is related to a particular aspect of the alternatives. For example, when planning financial investments there may be different investment opportunities. In this case the components of x may denote the amount of money invested in each opportunity.

Usually not all the points in the space defined by the components of x—called the *decision space*—represent a feasible alternative. Normally certain constraints exist restricting the feasible set X to a subset of the decision space. In financial planning, the constraints may result from restricted budgets, or from certain conditions concerning structural relations between different investments. The use of constraints in decision space yields an implicit definition of the feasible set, because the individual alternatives are not explicitly known. From a decision-theoretic point of view, there is no major difference between an explicit or implicit definition of the feasible set. However, in the latter case there is the additional problem of identifying feasible alternatives. On the other hand, an implicit formulation may provide a structure that can be exploited by special solution techniques.

1.1.2. Criteria and Outcomes

The decision maker has to define a set of *criteria* that reflects the various consequences arising from the choice of an alternative. In MCDM more than one consequence is considered, which means that every alternative has an image in an m-dimensional *outcome space*. For each of the relevant consequences a function c_i (i=1,…,m) has to be defined. Based on the functions c_i, a vector-valued function $c(x):=(c_1(x), c_2(x),\ldots, c_m(x))$ can be defined. The vector $y=c(x)$ is termed *outcome of x*. The image of the feasible set X under the function c is termed the *set of feasible outcomes*, and is denoted by Y. Typically $\Re^m$ is used as the outcome space, which means that each criterion is a real-valued function $c_i: X\rightarrow\Re$, leading to the following definition

$$c: X \rightarrow Y \subset \Re^m$$

$$y = c(x) = (c_1(x), c_2(x), \ldots, c_m(x)) \tag{1}$$

It should be mentioned that the assumption of real-valued criteria is not a restrictive one. In fact, this is only a question of encoding. For instance, for the criterion “color” the domain of possible values such as “black,” “white,” “red,” and so on may be encoded by real numbers.

In a real decision situation, defining a set of criteria may be a crucial task. The following requirements should be fulfilled for the set of criteria considered:

- *Completeness:* all aspects that must be evaluated by the decision maker should be covered by the system of criteria. A necessary condition for completeness is that the decision maker has to be impartial with regard to any pair of alternatives with identical outcomes.

- *Mutual exclusiveness:* each criterion should only measure aspects of the problem that are not measured by any other criterion. This is to avoid double counting of aspects.

- *Reliability:* each criterion should assess precisely the aspect it is intended to measure.

- *Appropriate precision:* each criterion should assess the aspect it is intended to measure as precisely as necessary.

- *Independence:* different types of independence are considered in literature. The fundamental type is known as *weak preference independence*. A criterion is called weakly preference-independent of the other criteria if its evaluation is independent of the values of all other criteria. For example, the criterion “color” for evaluating a car is *not* weakly preference-independent if a white Rolls Royce is preferred to a red one, but a red Ferrari is preferred to a white one.

- *Non-redundancy:* for reasons of economy, the system of criteria should be as small as possible. A criterion is redundant if its deletion does not affect the comparative evaluation of any pair of alternatives by the decision maker.

In complex situations with many criteria, a hierarchy is often formulated. In such a hierarchy, the general criteria on the higher levels are gradually broken down into more specific ones on the lower levels.

Another important issue is the question of how a criterion measures the decision maker’s preferences. Figure 2 shows three different types of measurement scales.

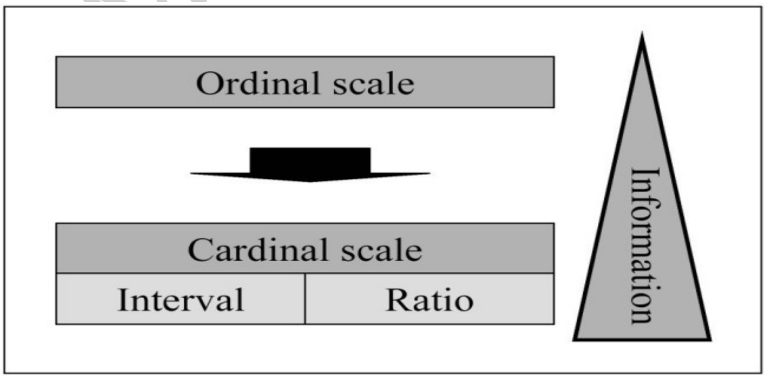

Figure 2. Hierarchy of scales

On an *ordinal scale,* there is an ordering of the criterion values. When using real-valued criteria, the natural ordering of real numbers can be used to formulate both maximizing and minimizing problems. It should be pointed out that the numbers on an ordinal scale provide no information about strength of preference: they only define a rank order for the set of alternatives.

On a *cardinal scale,* a criterion provides more information about the decision maker's preferences than it does on an ordinal scale. The criteria values do not only imply a rank order; they also reflect the strength of preference between any pair of them. Two types of cardinal scales, *interval* and *ratio scale*, can be distinguished. On an interval scale, the strength of preference is measured by the difference between the values of the criteria, whereas on a ratio scale it is measured by the quotient of these values. If a criterion is measured on a ratio scale, the following statement is possible: "concerning this criterion, A is twice as good as B." On an interval scale a typical preference statement is: "concerning this criterion, the preference of A over B is greater then the preference of C over D." It should be noted that, because of the increase of preference information obtained by changing from ordinal to cardinal scales, the assessment of the criteria in the latter mode is more demanding.

In a decision problem with a finite feasible set, one can arrange all information concerning alternatives and criteria in a so-called *decision matrix*. For the house-purchase example, the decision matrix is given in Table 1.

Criterion Alternative	**c_1: Number of Rooms**	**c_2: Condition**	**c_3: Age [year]**	**c_4: Price [$]**	**c_5: Distance to Center [miles]**
x_1: Ash Street	10	good (4)	4	260,000	5
x_2: Beacon Avenue	11	very good (5)	5	240,000	4
x_3: Cambridge Street	7	poor (2)	15	200,000	7
x_4: Davis Square	7	very poor (1)	15	200,000	8
x_5: Exeter Road	7	very poor (1)	20	220,000	8
x_6: Forest Street	9	fair (3)	10	240,000	6
x_7: Glen Road	13	very good (5)	0	320,000	3

Table 1. Decision matrix for the house-purchase example

In this example there are seven alternatives, x_1 to x_7, evaluated by five criteria, c_1 to c_5. Obviously, the second criterion "condition" is qualitative in nature. In brackets, a numerical encoding is given which translates it into a maximizing criterion on an ordinal scale. All the other criteria are already numerical, and can be measured on a cardinal scale. Furthermore, it is assumed that the first two criteria should be maximized while the others should be minimized. Note that the rows of this decision matrix represent the outcomes of the alternatives. The set of feasible outcomes consists of seven vectors, and is a subset of the five-dimensional outcome space defined by the dimensions c_1 to c_5.

1.1.3. Preferences

In MCDM it is assumed that the decision maker takes a decision by looking not at the alternatives directly but at their outcomes. Therefore the decision maker's *preferences* are defined in outcome space. For the house-purchase example, this means that the decision maker's preferences do not depend on the houses as such but on their price, size, situation in residential areas, and other factors. It is assumed that a binary relation $y \succsim y'$ exists, defined on the set of feasible outcomes often referred to as *weak preference,* with the meaning *y is at least as good as y'*. This relation may be separated into its symmetric and asymmetric parts, termed *indifference* (~) and *strict preference* ($\succ$).

Generally, the following assumptions are made: Strict preference is asymmetric ($y \succ y' \Rightarrow y' \nsucc y$) and transitive ($y \succ y' \wedge y' \succ y'' \Rightarrow y \succ y''$). Indifference is reflexive ($y \sim y$), symmetric ($y \sim y' \Rightarrow y' \sim y$), and transitive ($y \sim y' \wedge y' \sim y'' \Rightarrow y \sim y''$). Note that symmetry of indifference is directly related to the completeness of the system of criteria. Furthermore, it is assumed that there is only partial knowledge about this preference relation, at least at the beginning of the decision process.

One main characteristic of the different approaches to MCDM is concerned with additional assumptions about the properties of the decision maker's preference relations. The most important one is the assumption of completeness of the preference relation, implying a complete pre-ordering of the set of feasible outcomes. The existence of a complete pre-order is a rather strong assumption, since it implies that the decision maker is able to give a valid preference statement on any pair of feasible outcomes.

1.1.4. Decisions

A decision problem can appear in various different ways, but three types are most often found:

- Choice problems
- Sorting problems
- Ranking problems

A *choice problem* is the type of decision problem one thinks of most often when considering decision making. It is the problem of selecting a "small" subset X* of "best solutions" from the feasible set X: in other words, identifying $X^* \subset X$. In most cases, only one alternative must be chosen. The house-purchase example represents such a typical choice problem if only one house is to be bought.

A *sorting problem* is the task of partitioning the feasible set into some subsets X_i^*: in other words the task is to identify $X_1^*, X_2^*, \ldots, X_k^* \subseteq X$, so that the X_i^* do not have any element in common and $\cup X_i^* = X$. Problems of this type appear, for instance, at a bank making decisions on whether to give loans to clients. Here, the set of applicants is often separated into three subsets. The first is defined as the set of applicants who will definitely *not* receive a loan. The second subset contains those applicants who will receive their loan. The final subset will be formed of those applicants who need further consideration. Sorting problems often arise in some phase of a more complex decision process. For instance, in the house-purchase example one may first subdivide the houses found in the newspaper into three classes. The data available on the houses in the first class is sufficient to consider them for the final choice. The houses in the second class might also be interesting, but more information is needed to confirm this. The houses in the third class are not attractive and will not be considered further.

Finally, in a *ranking problem* achieving a rank order of the alternatives is the desired result. Problems of this type appear, for instance, when filling an academic position and a list of candidates must be proposed. Again, this problem may also be a phase of a more complex decision process. In the house-purchase example, a rank order of the available houses could be defined first before starting negotiations with the owners, one after another, according to this order.

From a mathematical point of view, sorting and ranking problems can be reformulated as choice problems. For instance, a sorting problem can be transformed into a choice problem by defining the feasible set as the set of all partitions of the original set of alternatives. Accordingly, for a ranking problem the feasible set can be defined as the set of all rankings of the alternatives. Since the choice problem with exactly one alternative representing the final decision is the most common type of decision problem, the material that follows focuses on this type.

1.2. Dominance and Efficiency

In MCDM, the concept of optimality is adapted to the multi-dimensionality of the outcome space by the concept of *dominance.* Assuming, without loss of generality, that all the criteria are minimizing criteria, dominance can be defined as follows:

Definition: let $y=c(x)\in Y$ and $y'=c(x')\in Y$ be feasible outcomes and $y\neq y'$. *y dominates y'* if, and only if, $y_i\leq y_i'$ for all $i\in\{1,2,\ldots,m\}$. Accordingly, in this case we say of the alternatives that *x dominates x'*.

In choice problems with exactly one alternative representing the final decision, the concept of dominance may be used to discard some alternatives (the dominated ones) from consideration. If the set of criteria is complete and there is an alternative x dominating some other alternative x' it would not be reasonable to choose the dominated alternative x', since x is at least as good as x' with respect to every criterion. On the contrary, if the decision maker prefers alternative x' to its dominating alternative x, then the system of criteria must either be incomplete or ill-defined.

Strongly related to the dominance is the concept of *efficiency*, as defined below.

Definition: an alternative $x\in X$ is called *efficient* if, and only if, no other feasible alternative $x'\in X$ dominating x exists: in other words, if $c_i(x')\leq c_i(x)$ holds for all $i\in\{1,2,\ldots,m\}$ then $c(x')=c(x)$. The set of efficient alternatives is called the *efficient set*, denoted by $Eff(X)\subseteq X$. Analogously the set of non-dominated outcomes in the outcome space, called the *set of efficient outcomes,* is denoted by $Eff(Y)\subseteq Y$. In fact, it is the image of the efficient set under the function c.

Due to the importance of the concept of efficiency, other types of efficiency have also been discussed in the literature. It should be pointed out that the concepts of dominance and efficiency require all criteria to be measured using ordinal scales at least.

Example: looking at the house-purchase example and taking into account that the first two criteria are to be maximized, while the others are of the minimizing type, it can be seen that "Beacon Street" dominates "Forest Street," "Cambridge Street" dominates "Davis Square," and "Davis Square" dominates "Exeter Road". Moreover, through the transitivity of the dominance relation it follows that "Cambridge Street" also dominates "Exeter Road." Since these are the only dominance relations, it follows that the efficient set consists of "Ash Street," "Beacon Avenue," "Cambridge Street," and "Glen Road."

1.3. Basic Approaches

Apart from in the extraordinary situation when an alternative x* dominates all others exists (i.e. Eff(X)={x*}), the concept of efficiency cannot be used to solve a choice problem. The efficient set may contain a number of quite different alternatives, with outcomes widespread over the outcome space. However, under the assumption that the preferences of the decision maker are captured by preference relations that form a complete preorder in outcome space, a unique best alternative (or set of equal-best alternatives) exists. The main task in MCDM is bridging the gap between the efficient set and a best alternative according to the decision maker's preferences. There are two fundamentally different approaches to this. The first standard one, termed the *value function approach*, tries to close this gap by representing the decision maker's preferences using a real-valued function on the outcome set, called a *value function*. This approach is inspired by the fact that the existence of a complete preorder on the outcome set, together with some continuity assumptions of little practical importance, implies the existence of a value function representing the decision maker's preferences. Moreover, this is a very elegant approach from a methodological point of view since it transforms the multiple criteria problem into a standard, single-criterion optimization problem. Given a value function, solving the problem can be an algorithmic problem, but is no longer a decision-theoretical one. The key problem associated with this approach is the determination of the value function, which will be discussed in the next section. It should be mentioned that the term "value function" is used here according to the more recent decision theory literature, while the term "utility function" is applied to the analogue approach under risk. The other approach to finding a best alternative in a multiple-criteria decision problem is the *vector optimization approach*. Here one tries to avoid constructing a detailed model of the decision maker's preferences. In an interactive process, the decision maker is provided with information about the outcome set and is asked for some preference information. Organizing this process of exchanging information iteratively, a search process in the efficient set is realized.

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Biographical Sketches

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