MEASUREMENT OF RISK

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Contents

Summary

The concept of risk is essential to many problems in economics and business. Usually, risk is treated in the traditional expected utility framework where it is defined only indirectly through the shape of the utility function. The purpose of utility functions, however, is to model preferences.

In this article, those approaches are reviewed which directly model risk judgements. After a review of standardized risk measures, recent theoretical developments of measures of perceived risk are presented.

1. Introduction

The term *risk* plays a pervasive role in many economic, political, social, and technological issues. In the literature, there are various attempts to define or to characterize the risk of an alternative for descriptive as well as for prescriptive purposes. Thereby, the main emphasis lies on the risk itself of the alternative, independently of the problem of risk preference. *Risk* refers to the riskiness of an alternative. It is a matter of perception or estimation. *Risk preference* refers to the preferability of an alternative under conditions of risk and is a matter of preferences.

Having accepted that risk is something different from risk preference, it would be interesting to know what the relation between risk and risk preference is. There are various theories of decision making under risk. Some of these theories like risk-value models make explicit use of a risk measure, others do not. In this article, *neither* risk-value models *nor* the relation of risk measures to other theories of decision making under risk will be discussed. The focus is on one important component of risk-value models, the factor risk, independently of any risk preferences.

There are three main reasons, which necessitate a means for the direct comparison of alternatives as to their risk. First, the understanding of riskiness judgements might help to understand preference. Taking risk and value as primitives, one might explain preference by a risk-value model, i. e. by a function of these two components. Many theories in management and finance rely on such a separate consideration of risk and value. Possibly, the best known example is modern portfolio selection theory. Within this context, the decision problem is viewed as choosing among possible risk-return combinations and formulated as either maximizing return for a given level of risk or minimizing risk for a given level of return. With such an approach, obviously, the decision will generally depend on the risk measure used. Second, there is growing empirical evidence that, under conditions of uncertainty, people base their decisions on qualitative aspects of choice alternatives such as risk. Finally, judgements of perceived risk may be required as such, independent of the necessity of choice, e. g., for intervention before the decision stage in a public policy setting.

In this article, it is started from the assumption that there exists a meaningful *risk ordering* which can be obtained directly by asking an individual to judge which of any pair of comparable alternatives is riskier. The key concept will therefore be a binary relation $\succsim$, with $A \succsim B$ meaning that an alternative *A* is at least as risky as another alternative *B*. Throughout the article the relation $A \succ B$ states that alternative *A* is riskier than alternative *B* while *A* ~ *B* means that *A* and *B* are equally risky. The risk ordering need not be related to the individual's preference ordering in any simple way. According to the conception of standard measurement theory, functions *R* are searched for which numerically represent the relation $\succsim$, i. e. functions *R* with the property

$$A \succsim B \quad \Leftrightarrow \quad R(A) \geq R(B)\,. \tag{1}$$

Every such function *R* will be called *risk measurement function* or simply *risk measure*.

Despite the importance of risk, there is little consensus on its definition. In empirical studies, typically, two dimensions, which appear to determine perceived risk, have been identified: amount of potential loss and probability of occurrence of loss. The risk of an alternative increases if the probability of loss increases or if the amount of potential loss increases. Unfortunately, up to now no agreement has been reached on the relative importance of the uncertainty of outcomes versus their undesirability for determining perceived risk. Furthermore, there is empirical evidence that possible gains reduce the perceived risk of an alternative. However, it is by no means clear how and to what extent risk perception depends on potential gains. Thereby, losses and gains are defined with reference to a certain target outcome. This target outcome may be the zero outcome, status quo, a certain aspiration level, as well as the best result attainable in a certain situation. An outcome is regarded as a loss if and only if it falls below the target outcome. It is regarded as a gain if and only if it lies above the target return. Other empirical studies have shown that risk is not simply equal to something like negative preference, it is an own important concept. When judging the riskiness of an alternative, people encode and combine probability and outcome information in qualitatively different ways than when judging its attractiveness.

Risk measurement as it is presented in this article seeks to get behind specific contextual referents of risky alternatives to consider characteristics of risk that apply to many different situations. It is the objective of this article to review the more naive standardized risk measures as well as recently developed economic or psychological theories of perceived risk which rely on the axiomatic approach of modern measurement theory.

2. Standardized Risk Measures

In this section, an overview is given on measures of risk, which have been advanced to quantify risk in a standardized way, which is widely acceptable and independent of individually varying perception. Among all the measures reviewed, subjective transformation of values or probabilities is not admitted.

Traditionally, the risk of an alternative has primarily been associated with the dispersion of the corresponding random variable of monetary outcomes. Then, it is common to measure the riskiness of an alternative by its *variance* σ^2 or its *standard deviation* σ. If an alternative's future value is characterized by a continuous random variable $\overline{x}$ with density $f = f_{\tilde{x}}$, distribution $F = F_{\tilde{x}}$, and expectation

$$\mu := E(\tilde{x}) := \int_{-\infty}^{+\infty} x f(x) dx \tag{2}$$

these risk measures are defined by

$$\sigma^2 := \mathrm{Var}(\tilde{x}) := \int_{-\infty}^{+\infty} (x-\mu)^2 f(x) dx \tag{3}$$

and

$$\sigma := \left[\int_{-\infty}^{+\infty} (x-\mu)^2 f(x) dx \right]^{1/2} . \tag{4}$$

In the finance context, the standard deviation of continuous growth rates usually is called *volatility*.

Similar standardized risk measures are the *expected absolute deviation around μ*

$$\int_{-\infty}^{+\infty} |x-\mu| f(x) dx \tag{5}$$

and the *expected absolute deviation around* 0

$$\int_{-\infty}^{+\infty} |x| \; f(x)dx. \tag{6}$$

In the context of technological issues, the risk of a project sometimes is simply quantified by the product

$$x \cdot p(x), \tag{7}$$

where x is the costs of some "catastrophic" event connected with the project and $p(x)$ its corresponding probability. In fact, this "measure" is a gross simplification of (6).

Besides, it has been conventional wisdom in economics and other fields of research that risk is the chance of something bad happening. In this vein, risk is associated with an outcome that is worse than some specific target outcome and its probability. Within the risk measures tailored to this notion of risk are the *lower semivariance*

$$\int_{-\infty}^{\mu} (x-\mu)^2 \, f(x)\, dx, \tag{8}$$

the *expected value of loss*

$$-\int_{-\infty}^{0} x \;\, f(x)dx, \tag{9}$$

and the probability of loss or probability of ruin

$$P_{\tilde{x}}(\tilde{x} \le r) = \int_{-\infty}^{r} f(x)dx. \tag{10}$$

Thereby, r is a certain target level outcomes lower of which are a loss or disastrous to the decision maker.

In the same vein in 1977, Fishburn proposed the risk measure

$$R_F(\tilde{x}) = \int_{-\infty}^{t} (t-x)^k \; f_{\tilde{x}}(x)dx \qquad (k > 0). \tag{11}$$

Thereby, t is a fixed upper bound, $t \le E\tilde{x}$. The parameter k of this risk measure may be interpreted as a risk-parameter characterizing a kind of risk attitude. Values k > 1 describe a certain risk-sensitive, values $k \in (0, 1)$ a certain risk-insensitive behavior.

Fishburn's risk measure can be interpreted as a certain moment of the distribution of $\tilde{x}$. As the lower semivariance, it is a 'lower moment' characterizing the part of the distribution below the expectation. It is 'partial' because this part is only partially characterized. Because this characterization is relative to the parameter t, Fishburn's risk measure simply constitutes what in the literature now is called the *lower partial moment (relative to t)* of the distribution of $\tilde{x}$.

To measure the market risk of a portfolio of traded assets, banks are more and more employing internal models based on a methodology called *Value-at-Risk*. This methodology serves for the determination of the capital requirements that banks have to fulfill in order to back their trading activities. For a given time horizon and a confidence level $1 - \alpha$, the Value-at-Risk of a portfolio is the loss in market value over the time horizon that is exceeded by the portfolio only with probability α.

Let r be the reference level with which the value of a given portfolio is compared at the end of the time horizon. If $x < r$, there is a loss at the amount of $r - x$. The portfolio's loss is thus given by the random variable

$$\tilde{l} := r - \tilde{x} \tag{12}$$

As reference level, initial value x_0 as well as expected value $E(\tilde{x})$ may reasonably be used. The probability of a loss lower than or equal to l is given by the distribution function

$$F_{\tilde{l}}(l) := P(\tilde{l} \le l) = \int_{-\infty}^{l} f_{\tilde{l}}(t)dt\,. \tag{13}$$

Using the loss distribution $F_{\tilde{l}}$, for a given time horizon and a given confidence level $1-\alpha\left(0 \le \alpha \le 1, e.g.\, \alpha = 0.01\right)$, the $p \cdot 100\%$ *Value-at-Risk* of the portfolio is the loss $\mathrm{VaR} = \mathrm{VaR}_{\tilde{x};\alpha}$ implicitly defined by

$$F_{\tilde{l}}(VaR) = P(\tilde{l} \le VaR) = 1 - \alpha\,. \tag{14}$$

This equation shows that, statistically speaking, the VaR-measure of a portfolio is the $1-\alpha \cdot 100\%$ - *quantile of the portfolio's loss distribution*.

Applying the inverse distribution function $F_{\tilde{l}}^{-1}$ to (14) yields the $1-\alpha \cdot 100\%$ Value-at-Risk of the portfolio explicitly through

$$VaR = VaR(\tilde{x}) := F_{\tilde{l}}^{-1}(1-\alpha)\,. \tag{15}$$

Thereby, $F_{\tilde{l}}^{-1}(1-\alpha)$ is the value of the inverse distribution function $F_{\tilde{l}}^{-1}$ at $1-\alpha$.

All the risk measures reviewed above are special cases of two related three-parameter families of risk measures. The first three-parameter risk measure is defined as

$$R_{S1}(\tilde{x}) := \int_{-\infty}^{q(F_{\tilde{x}})} \left| x - p(F_{\tilde{x}}) \right|^k dF_{\tilde{x}}(x) \quad (k \geq 0)\,, \tag{16}$$

where $p = p(F_{\tilde{x}})$ denotes a reference value level from which deviations are measured. The positive number k specifies a power to which deviations in value from the reference level are raised and thus k is a measure of the relative impact of large and small deviations. The parameter $q = q(F_{\tilde{x}})$ is a range parameter that specifies what deviations are to be included in the risk measure. The second three-parameter risk measure is defined to be the k^{th} root of $RS1(\tilde{x})$, i.e.,

$$R_{S2}(\tilde{x}) := \left[\int_{-\infty}^{q(F_{\tilde{x}})} \left| x - p(F_{\tilde{x}}) \right|^k dF_{\tilde{x}}(x) \right]^{1/k} \quad (k > 0)\,. \tag{17}$$

Through appropriate choices of the parameters $p = p(F_{\tilde{x}}), q = q(F_{\tilde{x}})$, and k it is easy to see that the above reviewed risk measures are special cases of one of the families (16) and (17). The variance (3) results from equation (16) and the standard deviation (4) from equation (17) by setting $p = E(\tilde{x}), k = 2$ and $q = +\infty$. The expected absolute deviation around μ and around 0, (5) and (6) respectively, are special cases of (16) obtained by choosing $p = \mu = E(\tilde{x})$ and $p = 0$, respectively, $k = 1$, and $q = +\infty$. Equation (16) gives the lower semivariance (8) when $k = 2$ and $p(F_{\tilde{x}}) = q(F_{\tilde{x}}) = \mu$; it gives the expected value of loss (9) when $p = q = 0$, and $k = 1$. Family (16) amounts to the probability of loss (10) by setting $k = 0$, and $q = r$. Finally, this family yields the lower partial moments (11) by setting $p = q = t$.

For any triplet (p, q, k) of parameter values, through both of these families of risk measures the risk of a given alternative is characterized by a nonnegative number $R = R(p, q, k)$. In fact, through both of these families of risk measures, the risk of a given alternative is characterized by a quadruplet (p, q, k, R) where $R = R(p, q, k)$. Essentially, if any three of these four quantities are fixed the fourth quantity can be used as a (not necessarily nonnegative) real-valued indicator of the risk of a given portfolio. Based on that idea, additionally, it can be shown that also the Value-at-Risk is a special case of family (16).

For any of the risk measures reviewed so far, its embedding in one of the families (16) or (17) immediately discloses the features of this measure. It shows, e.g., that the variance indeed takes into account the idea of a target return, namely by choosing $p = E(\tilde{x})$, but that, by choosing $q = +\infty$, all deviations from that target return irrespective of being above or below the target return, symmetrically, are taken into account. In addition, outcomes above the target return increase the risk. This contradicts the empirical notion of risk outlined in the introduction.

This embedding also discloses the major features of the Value-at-Risk measure. It takes into account the idea of a target return by implicitly choosing $p = r$. Contrary to the variance and in the sense of the empirical notion of risk, only deviations from the target return downwards are considered. Another advantage of the Value-at-Risk measure is that by fixing the parameter α the risk of a portfolio is expressed in terms of value and is, therefore, easy to interpret. However, nevertheless, the Value-at-Risk is a very rudimentary risk measure. Because the parameter k is set to 0, obviously, it contains no information on the loss distribution. The Value-at-Risk user knows that a loss bigger than the Value-at-Risk will only happen with a certain (small) probability. He has no information on, e.g., how large a very big loss can be and how probable it is. Contrary to the empirical notion of risk, the Value-at-Risk measure does not increase if the amount of potential loss increases.

Other naive risk measures scattered in the literature are the *Shannon entropy*

$$\int_{-\infty}^{\infty} f(x)\ln(f(x))dx \tag{18}$$

which is well-known from communication theory, the *interquartile range* $F^{-1}(0.75) - F^{-1}(0.25)$, and the *minimum outcome* $x_{\min}$ of $\tilde{x}$. For cases where values $x < 0$, i. e. losses are possible, the minimum outcome is usually called *maximum loss*.

In the remainder of this article, an overview is given on economic or psychological theories of perceived risk. All the measures reviewed admit subjective transformations of values or probabilities.

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Biographical Sketch

Hans Wolfgang Brachinger is a Professor in the Department of Quantitative Economics at the University of Fribourg (Switzerland) and holds the Chair of Economic Statistics. After studies in Mathematics, Econometrics, and Statistics at the Universities of Munich, Regensburg, and Tübingen (Germany) he received a master’s degree in Mathematics at the University of Regensburg and doctoral degrees in Statistics and Econometrics of the University of Tübingen. His research areas are decision theory and economic statistics. Dr. Brachinger has published articles in scientific journals such as *The American Economic Review*, *Operations Research Spectrum, The Swiss Journal of Economics and Statistics,* as well as the *Journal of Economics and Statistics* and is contributor to the *Encyclopedia of Statistical Sciences* and the *Handbook of Utility*. He has been a referee for many scientific publications. He is a member of the *International Statistical Institute (ISI),* the *Institute for Operations Research and the Management Science (INFORMS)*, the *Decision Analysis Society (DAS), the Deutsche Statistische Gesellschaft (DStatG),* the *Deutsche Gesellschaft für Operations Research* (GOR) as well as of the *Gesellschaft für Wirtschafts- und Sozialwissenschaften - Verein für Socialpolitik* (VS) where he is fellow of the *Ausschuss für Ökonometrie* as well as of the *Sozialwissenschaftlicher Ausschuss.* He has been co-organizer and chairman of many national and international scientific conferences.