A SHORT HISTORY OF DYNAMICAL SYSTEMS THEORY: 1885-2007

Philip Holmes
Princeton University, Princeton, NJ 08544, U.S.A.

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Contents

Summary

This chapter reviews the early (1885 – 1975) and more recent (1975 – 2007) history of dynamical systems theory, identifying key principles and themes, including those of

dimension reduction, normal form transformation and unfolding of degenerate systems. Some recent extensions and applications are also sketched.

1. Introduction

Dynamical systems theory (also known as nonlinear dynamics or chaos theory) comprises a broad range of analytical, geometrical, topological, and numerical methods for analyzing differential equations and iterated mappings. As a mathematical theory, it should perhaps be viewed as a “normal” development within mathematics, rather than a scientific revolution or paradigm shift (cf. [Kuhn, 1970]), as some popular accounts have claimed [Gleick, 1987]. However, crucial motivations and ideas have entered this area of mathematics from the applied sciences, and a still-accelerating stream of applications driven by recent developments in dynamical systems theory began in the last third of the 20th century. See [Aubin and Dahan Dalmedico, 2002] for a “socio-historical” analysis which discusses such extra-mathematical influences and describes the confluence of ideas and traditions that occurred in Western Europe and the US in the turbulent decade around 1970.

This chapter provides a brief and possibly idiosyncratic survey of the development of dynamical systems theory from approximately 1885 through 1965-75, with some comments on more recent work and central themes that have emerged. I focus on finite-dimensional, deterministic systems: ordinary differential equations and iterated mappings; the important topic of ergodic theory [Katok and Hasselblatt, 1995] is mentioned only in passing. There is also a growing qualitative theory of stochastic dynamical systems, see, e.g. [L. Arnold, 1974, L. Arnold 1998]. This article is not a tutorial: technical details and precise statements are largely omitted, and the reader is referred to the many textbooks and monographs on dynamical systems, examples of which are cited in Section 5. Otherwise, the bibliography emphasizes original references, early review articles, and papers that were influential in directing the field (sometimes long after their publication). The mathematical viewpoint taken here does not intend to downplay important extra-mathematical motivations and contributions, some references to which are made, but to do them justice would demand a separate article.

2. The Qualitative Theory of Dynamical Systems

I start by declaring my belief that “chaos theory” lacks the status of, say, the quantum or relativity theories, and that “nonlinear science” is not a science in the manner of physics, chemistry or biology. Dynamical systems theory neither addresses specific phenomena nor proposes particular models of (parts of) reality; it is, rather, a loosely-related set of *methods* for analyzing ordinary differential equations (ODEs) and iterated mappings, the canonical problems addressed having the forms:

$$\dot{x}_j = f_j\left(x_1, x_2, \ldots, x_n; \mu_1, \ldots, \mu_k\right), \quad \text{or} \quad x_j\left(l+1\right) = F_j\left(x_1(l), \ldots, x_n(l); \mu_1, \ldots, \mu_k\right) \qquad (1)$$

where $x_1, \ldots, x_n$ are *state variables* and $\mu_1, \ldots, \mu_k$ are external *control parameters*, usually regarded as fixed for the purpose of solving (1) to obtain *orbits* $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ or $\left\{\mathbf{x}(l)\right\}_{l=0}^{\infty}$. The methods emphasize the study of *all* orbits or solution curves of (1), the dependence of the set of solutions or *phase portrait* on the parameters, and the description of *qualitative* properties such as the existence of periodic solutions, rather than derivation of explicit closed form expressions or approximations. Very few nonlinear ODEs or maps can be solved exactly, but as noted below, much can be deduced from the study of systems linearized around fixed points or other invariant sets. For example, the linearization of the ODE of (1) about a *fixed point* $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$, where $f_j(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) = 0$ for $j = 1, \ldots, n$, is

$$\dot{\xi} = \mathbf{Df}(\overline{\mathbf{x}})\xi \tag{2}$$

where $\mathbf{Df}(\overline{\mathbf{x}})$ is the $n \times n$ Jacobian matrix of first partial derivatives of the *vector field* $\mathbf{f} = (f_1, f_2, \ldots, f_n)$, evaluated at $\overline{\mathbf{x}}$. This constant coefficient linear system is easily solved by assuming an exponential form $\xi = \mathbf{v} \exp \lambda t$ and computing the eigenvalues λ_j and eigenvectors $\mathbf{v}_j$ of $\mathbf{Df}(\overline{\mathbf{x}})$. Thus the power of linear analysis can be locally brought to bear on a nonlinear system.

The qualitative theory of dynamical systems (to give it its full title) is a *mathematical theory* largely built on the pillars of analysis, geometry, and topology, the first and last of which, in turn, had their origins in Newtonian mechanics. While claims of scientific revolution may be exaggerated, the increasing reach of dynamical systems theory beyond the mathematical sciences is a fact. Its goal of classifying dynamical systems provides a unifying structure across a wide range of applications: it does not help one *formulate* models *per se*, but knowing the nature of some creatures in the mathematical jungle is clearly useful. However, while the theory might allow one to prove that models of the solar system, weather, or the world economy are chaotic (or stable), the validity of conclusions drawn regarding asteroid impacts, hurricanes or stock-market crashes will depend on the quality of the models themselves.

2.1. Early History: Homoclinic Points and Global Behavior

In naming the next three subsections after five major contributors to the theory, I wish primarily to provide a mnemonic device, not to denigrate the collaborative process of mathematical discovery and invention.

2.1.1. Poincaré and Birkhoff

The modern theory of dynamical systems derives from the work of H.J. Poincaré (1854-1912) on the three-body problem of celestial mechanics [Poincaré, 1892, 1893, 1899], and primarily from a single, massive and initially-flawed paper [Poincaré 1890]. In this paper, which won a prize celebrating the 60th birthday of King Oscar II of Sweden and Norway, Poincaré laid the foundations for qualitative analysis of nonlinear differential equations, and began to develop a coherent set of mathematical tools for their study. His paper describes the use of first return (Poincaré) maps for the study of periodic motions; defines *stable and unstable manifolds*; discusses stability issues at length, develops perturbation methods, and includes the (Poincaré) recurrence theorem. Of particular relevance in the present context, while correcting the proofs, Poincaré realised that certain differential equations describing “simple” mechanical systems with two or more degrees of freedom were not integrable in the classical sense, due to the presence of “doubly-asymptotic” points, now called homo- and heteroclinic orbits. Moreover, he saw that these orbits had profound implications for the stability of motion in general, and that his previous belief that a version of the restricted three-body problem of celestial mechanics had only stable behavior, was false. In December 1889 and January 1990 he created a perturbation theory to detect what is now called chaos, and provided the first explicit example of it [Barrow-Green, 1997, Holmes, 1990].

Following Poincaré’s work, J. Hadamard (1865-1963) considered the dynamics of geodesic flows, but the next major thrust was due to G.D. Birkhoff (1884-1944), who early in his career had proved Poincaré’s “last theorem” on fixed points of annulus maps [Birkhoff, 1913]. In a book on two degree-of-freedom Hamiltonian systems [Birkhoff, 1927] that still bears reading, Birkhoff showed that, close to any homoclinic point of a two-dimensional mapping, there is a sequence of periodic points with periods approaching infinity. He subsequently proved that annulus maps having points of two distinct periods contain complex limit sets separating their domains of attraction [Birkhoff, 1932], thus providing a key clue for Cartwright and Littlewood in their studies of the van der Pol equation (see below).

2.1.2. Andronov and Kolmogorov

While Birkhoff was working in the U.S., A.A. Andronov (1901-1952), a student of L. Mandelstam (1879-1944), established a strong group in dynamics in the U.S.S.R at Gorki (Nizhni-Novgorod), and he and L.S. Pontryagin introduced the key idea of *structural stability* under the name “systèmes grossieres” (coarse systems) [Andronov and Pontryagin, 1937]. This notion, now a central theme of the theory, asks what properties are necessary and sufficient for the qualitative behavior of the flow comprising all solutions of a given ODE to survive a small perturbation to the vector field defining it. Here “survive” implies that the flows of the original and perturbed system must be topologically equivalent (homeomorphic). While subtle technical questions remain on function space topologies and norms that define “small” and “equivalent,” this approach launched the study of structurally stable systems, *and* of degenerate or *bifurcation* points at which arbitrarily small perturbations can produce qualitatively different behaviors [Andronov *et al.*, 1971]. The classification and *universal unfoldings* of such points provided for the first time lists of behaviors that one might expect when studying families of ODEs or maps depending on one or more (control) parameters. In the special case of gradient systems (vector fields derived from a potential function), this culminated in R. Thom’s (1923-2002) catastophe theory [Thom, 1975]. Thom’s book, and the work of Christopher Zeeman (b. 1925) [Zeeman, 1977] led the way in exporting previously-abstract mathematical ideas to the sciences at large, e.g. in [Chillingworth, 1976] and [Poston and Stewart, 1977], albeit not without controversy over the relevance of generic and universal ideas to specific problems in what were then new areas for mathematics [Sussmann and Zahler, 1978, Smale, 1980].

Andronov’s work was motivated by radio and electronic applications, and it led to further detailed studies of specific nonlinear oscillators, summarized in a classic text [Andronov, Vitt and Khaiken, 1966]. (Vitt’s name was missing from the title page of the first (1937) Russian edition, appearing only in the second edition with the enigmatic prefatory note that it had been omitted “by an unfortunate mistake.” The omission was due to Vitt’s death in a prison camp in Kolyma, Siberia in the winter of 1936-7.[1]) The book focused on planar ODEs, starting with conservative (Hamiltonian) systems, moving on to dissipative systems and discussing bifurcations of fixed points and limit cycles, including global (homoclinic) bifurcations. Unlike much of the Western mathematical literature, it contained explicit and practical examples. (For information on Andronov’s work in control theory and the Mandelstam school in mathematical physics, see http://ict.open.ac.uk/reports/1.pdf.) Following Andronov’s early death in 1952, his widow Leontovich continued to lead the Gorki group and two important early texts were produced [Andronov *et al.*, 1971, Andronov *et al*., 1973].

A more abstract approach was developed from the mid 1930’s in the “Moscow school,” gaining attention outside the U.S.S.R. via the translation of [Nemytskii and Stepanov, 1960], originally published in 1946, with an introduction by S. Lefshetz, who had himself written a key text a few years earlier [Lefschetz, 1957]. Here the first clearly-defined strange attractor – the solenoid – was described. The work of A.N. Kolmogorov (1903-1987), D.V. Anosov (b. 1936), V.I. Arnold (b. 1937) and Ya. G. Sinai (b. 1935) flourished in the 1950-60’s, largely in the context of Kolmogorov’s seminar. Important work was done on ergodic theory [Sinai, 1966], geodesic flows [Anosov, 1967] and billiards [Sinai, 1970], using Kolmogorov’s idea of K-systems. Some of this was motivated by S. Smale’s visit to Moscow in 1961, during which he met Anosov, Arnold and Sinai and told them of his discovery that structurally stable systems with infinitely many periodic orbits could exist (the Smale horseshoe: see below). While Russian mathematicians were somewhat isolated at the height of the cold war in the 1950-80s, and were rarely able to travel outside the Soviet Bloc, they eagerly sought contacts with the West, and visits such as those of Smale and Moser (see below), and the International Congress of Mathematicians held in Moscow in 1966 (which Smale also attended), helped maintain communications with European and American colleagues. Those able to visit Moscow found a hospitable welcome among their fellow mathematicians.

During this period M.M. Peixoto (b 1921) generalized the Andronov-Pontryagin results to flows on two-dimensional manifolds [Peixoto, 1962]. He proved that a flow on a compact two-dimensional manifold is structurally stable if and only if it has a finite number of fixed points and periodic orbits, all of which are hyperbolic, there are no orbits connecting saddle points, and the non-wandering set consists of fixed points and periodic orbits alone.

2.1.3. Smale’s Topological Viewpoint

S. Smale (b. 1930) brought topological ideas to these problems in the late 1950’s and began to generalize to $n > 2$ dimensions, defining gradient-like flows that are now called Morse-Smale systems. Such a flow (or map) has a finite set of fixed points and periodic orbits, all of which are hyperbolic and all of whose stable and unstable manifolds intersect transversely, but no other nonwandering or recurrent points. Smale

[1]I am indebted to an article by C. Bissell in the Times Higher Educational Supplement of January 28th, 1994 for this information.

conjectured that a system is structurally stable if and only if it is Morse-Smale. N. Levinson subsequently drew Smale's attention to a short paper on the periodically-forced van der Pol equation [Cartwright and Littlewood, 1945]. Levinson had worked on a simplified version of the problem [Levinson, 1949], and he suggested that it might provide a counterexample in the form of a structurally stable ODE with infinitely many periodic orbits. This led to Smale's creation of the *horseshoe map* in 1960, allegedly on the Leme beach of Rio de Janiero, while visiting Peixoto's Institute of Pure and Applied Mathematics. The story is recounted in [Smale, 1980], in the biography of [Batterson, 2000], and in [Diacu and Holmes, 1996]. The contributions of Cartwright and Littlewood are less well known: see [Cartwright, 1974, McMurran and Tattersall, 1996].

Smale's work appeared after a considerable delay [Smale, 1965], and became widely known only after an extensive survey article was published [Smale, 1967]. J. Moser (1928-1999) subsequently gave a beautiful exposition of the horseshoe [Moser, 1973], providing explicitly-testable criteria to prove its presence in two-dimensional maps and explaining clearly how the presence of dense orbits precludes the existence of additional integrals of motion. This implies that the problem for which King Oscar's prize was awarded is essentially insoluble. A pictorial account of the horseshoe can be found in [Shub 2005].

A footnote in [Cartwright and Littlewood, 1945] remarks that the authors' "faith in [their] results was at one time sustained only by the experimental evidence that stable sub-harmonics of two distinct orders did occur," referring to [van der Pol and van der Mark, 1927] and (implicitly) to [Birkhoff, 1932]. Smale was almost certainly ignorant of this work and of [Poincaré, 1890], but in proving that diffeomorphisms containing transverse homoclinic points possess nearby hyperbolic invariant sets on which the dynamics is conjugate to a shift on a finite alphabet of symbols, he completed the story that Poincaré had begun, connecting ODEs and deterministic maps with probabilistic Markov processes and showing that, in a deep sense, their orbits are indistinguishable. This is now referred to as the Smale-Birkhoff homoclinic theorem [Guckenheimer and Holmes, 1983]. Although significant progress was made on the piecewise-linear Levinson version [Levi, 1981] and the existence of strange attractors and invariant measures has been proved for related problems [Wang and Young, 2002], the van der Pol equation remains a research topic, e.g. [Guckenheimer *et al*., 2002, Bold *et al*., 2003].

2.1.4. Perturbations and Applications

Much of the work described above, including that of Peixoto and Smale, was topological in nature. Krylov and Bogoliubov developed analytical perturbation and averaging theories for nonlinear oscillation problems [Krylov and Bogoliubov, 1947]; these were employed extensively in Andronov's group, and generalized by [Hale, 1969] and others. Such methods were then used to prove the existence of transverse homoclinic orbits to periodic motions in periodically-forced oscillators [Melnikov, 1963] and in two- and three degree-of-freedom Hamiltonian systems [Arnold, 1964]. This provided the final link in a chain of methods and results that allows one to prove the existence of chaotic invariant sets in specific ODEs. Since then, "Melnikov's method" has been extended to multi- and infinite-dimensional systems including partial differential equations (PDEs) [Wiggins, 1988, Holmes and Marsden, 1981], and related ideas have been used to approximate Poincaré return maps near homoclinic orbits to equilibria and find similar chaotic sets [Silnikov, 1965]. In a return to the origins of Poincaré's work, it has been suggested that heteroclinic connections among unstable n-body orbits might provide routes for low energy space missions [Koon *et al*., 2000].

The work of Lorenz (b. 1917) on a three-dimensional ODE modeling Rayleigh-Bénard convection was done almost independently of that described above, although in presenting his discovery of *sensitive dependence on initial conditions*, [Lorenz, 1963] appealed to Birkhoff's work and (thanks to a perceptive reviewer) also to that of [Nemytskii and Stepanov, 1960]. But not until 1971, when Lorenz heard Ruelle speak on the proposal of [Ruelle and Takens, 1970] that structurally stable strange attractors might describe turbulence, were "dynamical" connections made between the meteorologists and mathematicians [Lorenz, 1993, Ruelle, 1991]. An earlier extra-mathematical discovery had taken place in 1961 when Y. Ueda, a graduate student in Electrical Engineering at Kyoto University, found motions that were neither periodic nor quasiperiodic in analog computer simulations of a periodically-forced van der Pol-Düffing equation. These solutions were mentioned as "complicated phenomena," in Ueda's PhD thesis, but remained otherwise unpublished until considerably later [Hayashi *et al*., 1970, Ueda *et al*., 1973], allegedly due to reservations of Ueda's advisor, C. Hayashi. For more on Ueda's work, see [Ueda, 2001].

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Biographical Sketch

Philip Holmes was born in North Lincolnshire, UK, in 1945. He holds a BA (Oxford, 1967) and a PhD (Southampton, 1974), both in engineering science. He is currently Professor of Mechanics and Applied Mathematics and an associate faculty member in the Department of Mathematics at Princeton University, where he directed the Program in Applied and Computational Mathematics from 1994 to 1997. Prior to that he was the Charles N. Mellowes Professor of Mechanics and Mathematics at Cornell University. A former director of Cornell's Center for Applied Mathematics, he is a Member of the American Academy of Arts and Sciences, a Foreign Member of the Hungarian Academy of Sciences, and a Fellow of the

American Physical Society. He has held visiting positions at The University of California at Berkeley, the Universities of Stuttgart, Nice and Paris, the Paul Erdos Mathematical Center in Budapest, and a Sherman Fairchild Distinguished Scholarship at the California Institute of Technology.

Dr. Holmes works on nonlinear dynamics and differential equations, developing qualitative and analytical methods for the mathematical modeling and study of solid, fluid, and biological systems. He has contributed to the foundations and applications of dynamical systems theory and has used the theory to better understand the dynamics of coherent structures in turbulent flows, pattern formation in chemical reactions, nonlinear modes in optical waveguides, and buckling and dynamics of elastic rods. His current research focuses on the neuromechanics of legged locomotion and swimming, and on modeling cognitive processes in humans and primates.