MEASURE THEORIES AND ERGODICITY PROBLEMS

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Summary

Measure concepts range among those mathematical theories having gone through a very long history. After the classical period they were often superseded by integration theories, to which they are obviously linked directly. But they never abandoned a specific pace. New horizons opened around 1932-1933 when Haar extended Lebesgue's integration theory over locally compact groups, and Kolmogoroff formalized Probability Theory rigorously.

The history of recent measure theories told in this chapter concentrates on abstract and concrete situations.

Ergodicity stemmed from rather different origins: The necessity of investigating qualitative solutions of differential equations recommended by Poincaré; the kinetic theory of gases pushed ahead by Boltzmann; the structure of surface transformations examined by Birkhoff. Nowadays ergodicity stands as an independent branch of mathematics, but also more and more connected to various other sciences.

1. Introduction

Since ancient time measures of lengths, surfaces, etc. have been performed with a view towards evaluations. The measurements should guarantee a certain permanence of the results, independent of transfers, in particular translations or rotations.

For Euclid, all objects which can be positioned to coincide, are equal.

Exploiting this principle it was possible for Greek mathematicians to obtain or verify, by *ad hoc* decompositions, lengths of segments, areas of polygons, volumes of polyhedra etc.

Later Greek methods were improved, thanks to tightening of exhaustion procedures, the methods developed by Gregorius of Saint-Vincent, Bonaventura Cavalieri, and forerunners of the invention of (infinitesimal) Calculus.

Of course Isaac Newton and Gottfried Wilhelm Leibniz are to be credited for having produced brand new ideas on the subject, for instance by performing integrations in view of the determination of areas.

Measure theories will often be naturally linked to integration theories, via the formal identification

$$\int 1_A(x)dm(x) = m(A)$$

with respect to a measure m and 1_A being the characteristic function of A, i.e., $1_A(x) = 1$ or 0 whether $x \in A$ or $x \notin A$.

The first widespread monograph on abstract measure theories was published by Paul Halmos in 1950; it presents a very comprehensive introduction to these themes and explains the fundamental properties leading to applications in mathematical analysis.

2. Measure Theories and Probability

In 1892 Camille Jordan defined an abstract measure on $\mathbb{R}^n$ relying closely on ancient Greek methods.

With respect to modern language a function is Riemann integrable if and only if its discontinuity points form a set of measure zero.

Hence the study of Riemann's integration theory raises the question about the importance of the size of the set of discontinuity points for a Riemann integrable function.

Otto Stolz and Carl Gustav Harnack introduced the pertinent ideas on $\mathbb{R}$. Georg Cantor extended the problem on $\mathbb{R}^n$. If E is a bounded subset of $\mathbb{R}^n$ and $\rho > 0$, let $V(\rho)$ be the neighborhood of E consisting of the points in $\mathbb{R}^n$ situated at a distance at most ρ from E. Now the 'measure' of E is the lower limit of the 'volumes' of $V(\rho)$.

The measure of any subset is the measure of its closure. So with respect to the definitions the measure of the union of two disjoint subsets may be strictly less than the sum of the measures of these subsets.

Cantor is also able to announce that an open subset of $\mathbb{R}$ is the countable union of mutually disjoint open intervals.

Émile Borel, not trying to approach a set from above, chooses for the measure of an open subset of $\mathbb{R}$ the sum of the lengths of its intervals. Then he introduces the class of the so-called Borel subsets: Starting from the class $\mathcal{O}$ of open subsets, he iterates countably many operations $U \cup V$ and $U \setminus V$, with $U, V \in \mathcal{O}$. To the whole class he attributes the 'complete additivity', the 'countable additivity', i.e., if a sequence is formed by pairwise disjoint Borel subsets, then the measure of the union of these subsets equals the sum of the measures of the subsets.

After the introduction of Borel sets Henri Lebesgue explains the motivations for his studies in his 1902 thesis.

"We want to attach to every bounded subset its measure satisfying the following conditions:

1. *There exist sets having measure nonzero.*
2. *Two sets that are equal [i.e. by transferring one of them, one may have them coinciding].*
3. *The measure of a finite or countable infinity of sets, without common points, is the sum of the measures of these sets.*

We are going to solve that measure problem only for the sets that we will call measurable" (1902, p. 255-256).

Actually Lebesgue had announced his results in a note to the *Comptes Rendus* of the French Academy of Sciences by April 29, 2001. That achievement may be considered to be the earliest important mathematical fact of the XX century.

In his 1904 *Leçons sur l'intégration et la recherche des fonctions primitives* Lebesgue imposes the following conditions on his integral for real-valued bounded functions on the line:

(1) For all *a, b, h,* one has $\int_a^b f(x)\,dx = \int_{a+h}^{b+h} f(x-h)\,dx$;

(2) For all a, b, c, one has $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^a f(x)\,dx = 0$;

(3) $\int_a^b [f(x)+\varphi(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b \varphi(x)\,dx$;

(4) If $f \geq 0$ and $b \geq a$, then $\int_a^b f(x) \geq 0$;

(5) $\int_0^1 1\,dx = 1$;

(6) If $f_n(x)$ converges increasingly to $f(x)$, the integral of $f_n(x)$ converges to the integral of $f(x)$.

The first five conditions are independent and Lebesgue wonders whether that could be the case also for (6). He observes that a Riemann integrable function satisfies (1) to (6) and the Riemann integral is then the only solution.

As it will suffice to consider characteristic functions, Lebesgue brings the problem down to the determination of a number $m(E) \geq 0$ for any bounded subset E, the measure of E, satisfying the following conditions:

(1’) Two subsets that are equal (i.e., by transferring one of them, it may be positioned to coincide with the other one) admit the same measure [Invariance of the measure];
(2’) The union of a finite or countably infinite number of pairwise disjoint subsets admits as its measure the sum of the measures of the subsets [countable additivity of the measure];
(3’) the measure of the interval $[0,1]$ is 1 [normalization of the measure].

In this setup (3’) replaces (5), (2’) comes out of (3) and (6), (1’) is (1). If $a < b$, the measure of the interval $[a,b]$ is $b-a$. An arbitrary bounded set E may be included into a finite or countably infinite number of intervals; the set of the sums of lengths of these intervals admits a lower limit called outer measure $m_e(E)$ of E. For a segment (bounded closed interval) containing E, $m(A)-m_e(A\setminus E)$ is the inner measure $m_i(E)$ of E. If $m_e(E)=m_i(E)$, the subset E is called measurable; the common value is the measure $m(E)$ of E verifying (1’), (2’), (3’).

All Borel sets belong to that class, but the latter is much wider. A Lebesgue measurable set is the union of a Borel set and a set of measure 0.

Borel had these comments on his own contribution:
“The sets for which one may define a measure by the preceding definitions will be called measurable by us, although we won’t claim that it is not possible to attribute measures to other sets; but such a definition would be useless; it could even be harmful in case it would not guarantee that the measure keeps the fundamental properties we have provided in our definitions” (1898, p. 48).

General abstract versions of Lebesgue’s procedure were performed by Johann Radon and Maurice Fréchet.

A measure space $(E,\mathcal{S},\mu)$ is defined for a set $\mathcal{S}$ of subsets in E which is closed with respect to unions, intersections, set subtractions, and a measure μ. For a measurable function f on E the inverse image of any measurable set must be an element of $\mathcal{S}$.

Bourbaki describes his views:
“The new theory [of Lebesgue] had still to be popularized, making it a mathematical instrument useful for common purposes, whereas the majority of mathematicians, around 1910, went on seeing in Lebesgue’s integral an instrument of very high precision, to be handled with care, to be used only for research of extreme subtlety and extreme abstraction. That was Carathéodory’s work, in a book remaining a classic for a long time, and did enrich Radon’s theory by numerous original remarks.

But with that book also for the first time the notion of integral is superseded by the notion of measure, that had been for Lebesgue just a technical help.” (1960, p. 257-258).

Carathéodory defines an outer measure on $\mathbb{R}^q$ via five axioms.

(I) The function μ^* is defined over all subsets of $\mathbb{R}^q$ and with values in $\overline{\mathbb{R}_+}$.
(II) If $B \subset A \subset \mathbb{R}^q$, then $\mu^*(B) \le \mu^*(A)$.
(III) If (A_n) is a finite or countably infinite sequence of subsets in R^q, one has $\mu^*(\cup_n A_n) \le \sum_n \mu^*(A_n)$ in case the latter series is convergent.

The subset A is called measurable if, for any subset W,

$$\mu^*(W) = \mu^*(A \cap W) + \mu^*(W \setminus (A \cap W));$$

in that case let $\mu(A) = \mu^*(A)$.

Carathéodory establishes a collection of fundamental properties deduced from the preceding definitions, stability properties of the class of measurable subsets.

He introduces a supplementary condition.
(IV) If $A_1, A_2 \subset \mathbb{R}^q$ and $\inf\{d(x,y); x \in A_1, y \in A_2\} > 0, d$ denoting the distance in $\mathbb{R}^q$, then

$$\mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2).$$

Finally the theory is completed by a last condition:
(V) The outer measure of a subset A is the lower limit of the numbers $\mu(B)$, B running through the set of measurable subsets containing A. The inner measure of A is defined by

$$\mu_*(A) = \mu^*(A) - \mu^*(B \setminus A),$$

where $A \subset B$. The subset A is measurable if and only if $\mu_*(A) = \mu^*(A)$.

As early as 1924, Alexander Khinchin, collaborator of Kolmogoroff, writes:
"The leading idea […] has been […] to range the fundamental concepts of probability theory, which so far had been considered to be quite specific, along a natural way in the row of the general formulations adopted for concepts in modern mathematics. Before the constitution of Lebesgue's measure and integral theory this task was quite hopeless. After Lebesgue's explorations the analogy between the measure of a set and the integral of a function as well as the mathematical expectation of a random object was at hand." (1924, p. iii).

During a first stage Andreï Kolmogoroff considers for the set E a set $\mathcal{S}$ of subsets in E which is closed with respect to unions, intersections, and set subtractions. In case $E \in \mathcal{S}$, the probability (measure) is a mapping $P: \mathcal{S} \to [0,1]$ such that $P(E) = 1$ and $P(A \cup B) = P(A) + P(B)$ for disjoint subsets A, B belonging to $\mathcal{S}$. Kolmogoroff claims also that for a decreasing sequence $(A_n) \in \mathcal{S}$ such that $\cap_n A_n = \emptyset$, one necessarily has $\lim_{n\to\infty} P(A_n) = 0$. For arbitrary subsets A, B in $\mathcal{S}$, $P(A \cap B) = P(A)P(B)$ holds.

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Biographical Sketch

Jean - Paul Pier, mathematician, is prof. em. of the Luxembourg Higher Education institution. He is the author of several books on mathematics and history of mathematics: *Amenable Locally Compact Groups* (Wiley, New York, 1984) *Amenable Banach Algebras* (Longman, 1988) *L'analyse harmonique, son développement historique* (Masson, Paris, 1990) *Histoire de l'intégration. Vingt-cinq siècles de mathématiques* (Masson, Pars, 1996) *Mathematical Analysis during the XX*$^{\text{th}}$ *century* (Oxford University

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