ECONOMIC DYNAMICS

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Contents

Summary

This chapter provides a comprehensive introduction of applications of differential and difference equations to economics. It introduces basic concepts and analytical methods and provides applications of these methods to solve economic problems. The chapter provides not only a comprehensive introduction to applications of theory of linear (and linearized) equations to economic analysis, but also studies nonlinear dynamic systems, which have been widely applied to economic analysis only in recent years. It is arranged according to dimensions of the dynamic systems. First, the chapter deals with scalar differential and difference equations. Next, it studies planar differential equations. Then, it introduces higher-dimensional equations. The chapter examines key concepts and main mathematical

results related to linear (linearized) and nonlinear equations and their applications to economics.

1. Introduction

The filed of applications of dynamic theory to economics is a vast and vibrant area. Concepts and theorems related to economic dynamics appear everywhere in academic journals and textbooks in economics. One can hardly approach, not to mention digest, the literature of economic theory and empirical studies about economic systems without proper knowledge of dynamic theory. This paper provides a comprehensive introduction of applications of differential and difference equations to economics. We introduce basic concepts and analytical methods and provide applications of these methods to solve economic problems.

Applications of differential and difference equations are now made in modeling motion and change in all areas of science. The dynamic theory has become an essential tool of economic analysis particularly since computer has become commonly available. The paper provides not only a comprehensive introduction to applications of theory of linear (and linearized) equations to economic analysis, but also studies nonlinear dynamic systems, which have been widely applied to economic analysis only in recent years. The paper is arranged according to dimensions of the dynamic systems. First, we deal with scalar differential and difference equations. Next, we study planar differential equations. Then, we introduce higher-dimensional equations. We examine key concepts and main mathematical results related to linear (linearized) and nonlinear equations and their applications to economics. Some examples are simulated with Mathematica. Today, more and more researchers and educators are using computer tools to solve – once seemingly impossible to calculate even three decades ago – complicated and tedious economic problems.

This paper is much based Zhang (2005, 2006). The lively pace of research on economic dynamics and their empirical applications to economics means that this paper cannot cover all the important applications of differential and difference equations to economic systems, not to mention current development of differential and difference equations, irrespective of the endeavors to provide a comprehensive study of the subject. We omit many other important fields of mathematics, such as partial differential equations, stochastic processes and dynamics with delay, which have also been applied to different fields of economics (Takayama, 1982, Weidlich and Haag, 1983, Puu, 2003, Caputo, 2005).

1.1. Differential Equations

A differential equation expresses the rate of change of the current state as a function of the current state. A simple illustration of this type of dependence is changes of the gross domestic product (GDP), $x(t)$, over time. The rate of change of the GDP is proportional to the current GDP itself

$$\dot{x}(t) = gx(t),$$

where t stands for time and $\dot{x}(t)$ the derivative of the function x with respect to t. The derivative is also represented by $dx(t)/dt$ or $x'(t)$. The growth rate of the GDP is $\dot{x}/x$. If the growth rate, g, is given at any point time t, the GDP at t is given by solving the differential equation. The solution is $x(t) = x(0)e^{gt}$. The solution tells that the GDP decays (increases) exponentially in time when g is negative (positive).

We can explicitly solve the above differential function when g is a constant. Nevertheless, it is reasonable to consider that the growth rate is affected by many factors, such as the current state of the economic system, accumulated knowledge of the economy, international environment, and many other conditions. This means that the growth rate may take on a complicated form, $g(x, t)$. The economic growth is described by

$$\dot{x}(t) = g(x(t), t)x(t).$$

In general, it is not easy to solve explicitly the above function. There are various established methods of solving different types of differential equations.

This chapter is mainly concerned with ordinary differential equations. Ordinary differential equations are differential equations whose solutions are functions of one independent variable, which we usually denote by t. Ordinary differential equations are classified as autonomous and nonautonomous. Equation $\dot{x}(t) = ax(t) + b$ with a and b as parameters is an autonomous differential equation because the time variable t does not explicitly appear. If the equation specially involves t, we call the equation nonautonomous or time-dependent. For instance, $\dot{x}(t) = x(t) + \sin t$ is a time-dependent differential equation. If an equation involves derivatives up to and includes the ith derivative, it is called an ith order differential equation. The equation $\dot{x}(t) = ax(t) + b$ with a and b as parameters is a first order autonomous differential equation. The equation, $\ddot{x} = 3\dot{x} - 2x + 2$, is a second order equation, where the second derivative, $\ddot{x}(t)$, is the derivative of $\dot{x}(t)$.

1.2. Difference Equations

A difference equation expresses the rate of change of the current state as a function of the current state. Consider the GDP of the economy in year t, $x(t)$, as the state variable in period t. Assume that the rate of change of the GDP, g, is constant. The motion of the GDP is then described by

$$\frac{x(t+1) - x(t)}{x(t)} = g,$$

where t stands for year. As the growth rate is given for each year, the GDP is given by solving the differential equation

$$x(t+1) = (1 + g)x(t).$$

In discrete dynamics, time, denoted by t, is taken to be a discrete variable so that the variable t is allowed to take only integer values. Different from the continuous-time dynamics, where the pattern of change of a variable, x, is embodied in its derivatives with respect to the change of t which is infinitesimal in magnitude, in the discrete dynamics the pattern of change of the variable must be described by "differences", rather than by derivatives of x. As the value of variable $x(t)$ will change only when the variable, t, changes from one integer value to the next, in difference equation theory t is referred to as period, in analytical sense.

To describe the pattern of change in x as a function of t, we introduce the difference quotient $\Delta x / \Delta t$. As t has to take integer values, we choose $\Delta t = 1$. Hence, the difference quotient $\Delta x / \Delta t$ is simplified to the expression Δx; this is called the first difference of x, and denoted by

$$\Delta x(t) \equiv x(t+1) - x(t).$$

We may express the pattern of change of x by, for instance

$$\Delta x(t+1) = -\, a x^{\alpha}(t).$$

Equations of this type are called difference equations. There are other forms of difference equations which are equivalent to the above forms, for instance

$$x(t+1) - x(t) = -\, a x^{\alpha}(t),$$

or

$$x(t+1) = x(t) - a x^{\alpha}(t) \equiv f(x(t)).$$

The evolution of the system starting from x_0 is given by the sequence

$$x_0, \;\; x(1) = f(x_0), \;\; x(2) = f(x(1)) = f(f(x_0)).$$

We usually write $f^2(x)$, $f^3(x)$, …, in place of $f(f(x))$, $f(ff((x)))$, …. Hence, we have

$$x(t+1) = f(x(t)) = f^{t+1}(x_0).$$

Here, $f(x_0)$ is called the first iterate of x_0 under f; $f^t(x_0)$ is called the tth iterate of x_0 under f. The set of all (positive) iterates $\{f^t(x_0) : t \ge 0\}$ where $f^0(x_0) = x_0$ is called the (positive) orbit of x_0 and is denoted by $O(x_0)$.

If the equation f is replaced by a function g of two variables, then we have

$$x(t+1) = g(x(t), t).$$

This equation is called nonautonomous or time-variant, whereas $x(t+1) = f(x(t))$ is called autonomous or time-invariant.

2. Scalar Linear Equations and Their Applications to Economics

2.1. Differential Equations

An initial value problem often consists of a differential equation together with enough initial values to specify a single solution. The solution of the initial problem

$$\dot{x}(t) = ax(t), \quad x_0 = x(0),$$

is $x(t) = x_0 e^{at}$. Figure 1 shows the family of solutions of differential equations for various initial values x_0 with $a < 0$ and $a > 0$. Each choice of initial value x_0 determines a curve. This picture is called flow of the differential equation. The flow, $\varphi(t, x_0)$, of an autonomous differential equation is the function of time t and initial value x_0, which represents the set of solutions. Thus $\varphi(t, x_0)$ is the value at time t of the solution with initial value x_0.

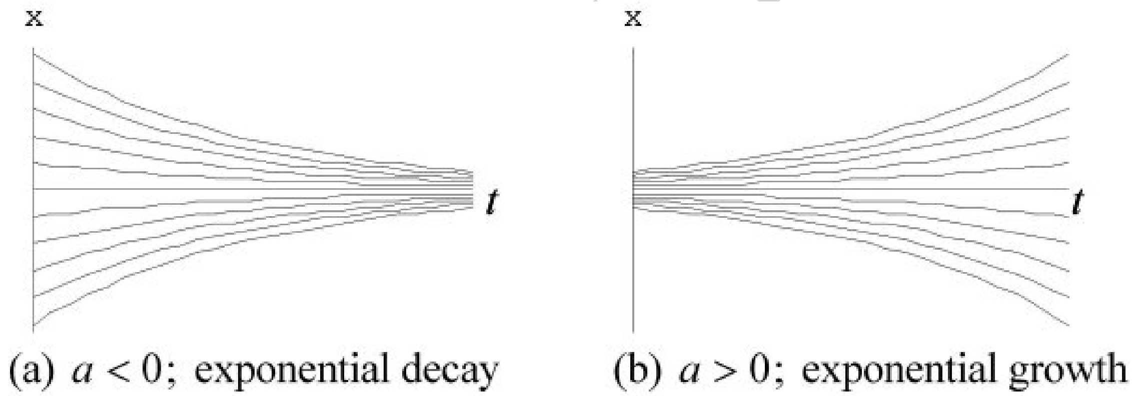


Figure 1: Solutions of $\dot{x} = ax$ with varied initial values

Linear first-order differential equations can be generally expressed

$$\dot{x} + u(t)x = w(t), \tag{1}$$

where $u(t)$ and $w(t)$ are functions of t.

The homogeneous case with constant coefficient and constant term

First, we examine the homogenous case of Eq. (1) when $u(t) = a$ and $w(t) = 0$. The solution of $\dot{x} + ax = 0$ is $x(t) = Ae^{-at}$, where A is an arbitrary constant.

The nonhomogeneous case with constant coefficient and constant term

A nonhomogenous linear different equation with constant coefficient is generally given by

$$\dot{x} + ax = b, \quad b \neq 0.$$

In the case of $a = 0$, the solution is $x(t) = bt + A$, where A is an arbitrary constant. In the case of $a \neq 0$, the solution with known initial state $x(0)$ is

$$x(t) = \left(x(0) - \frac{b}{a} \right) e^{-at} + \frac{b}{a}.$$

Definition 1: A constant solution of the autonomous differential equation $\dot{x}(t) = f(x)$ is called an equilibrium of the equation.

Example: We now consider dynamics of price of a single commodity. Suppose that the demand and supply functions for the commodity are

$$Q_d = a_1 - b_1 P, \quad Q_s = -a_2 + b_2 P, \quad a_j, b_j > 0, \tag{2}$$

where Q_d and Q_s are respectively the demand and supply for price P and a_j and b_j are parameters. Assume that the rate of price change with regard to time at t is proportional to the excess demand, $Q_d - Q_s$, that is

$$\dot{P}(t) = m(Q_d(t) - Q_s(t)), \quad m > 0.$$

Substituting Eqs. (2) into the above equation yields

$$\dot{P}(t) + m(b_1 + b_2)P = m(a_1 + a_2).$$

The solution of the equation is

$$P(t) = \left(P(0) - P^*\right)e^{-m_0 t} + P^*,$$

where

$$P^* \equiv \frac{a_1 + a_2}{b_1 + b_2}, \quad m_0 \equiv m(a_2 + b_2) > 0.$$

As m_0 is positive, we conclude that as $t \to +\infty$, $P(t) \to P^*$. In the long term, the market mechanism will lead the market dynamics to its equilibrium position.

The general case

$$\dot{x} + u(t)x = w(t),$$

where $u(t)$ and $w(t)$ are functions of t. The solution of this equation is

$$x(t) = e^{-\int u\,dt}\left(A + \int we^{\int u\,dt}\,dt \right),$$

where A is an arbitrary constant.

Example: the Solow-Swan model (Solow, 1956; Swan, 1956, see also Barro and Sala-i-Martin, 2004)

When the production function takes the Cobb-Douglas form, the dynamics of per-capita capital, $k(t)$, of the Solow-Swan model is given by

$$\dot{k} = sk^{\alpha} - \delta k, \;\; 0 < s, \delta, \alpha < 1,$$

where s, δ, and α are parameters. Introducing $z = x^{1-\alpha}$ transforms the model into

$$\dot{z} + (1-\alpha)\delta z = (1-\alpha)s.$$

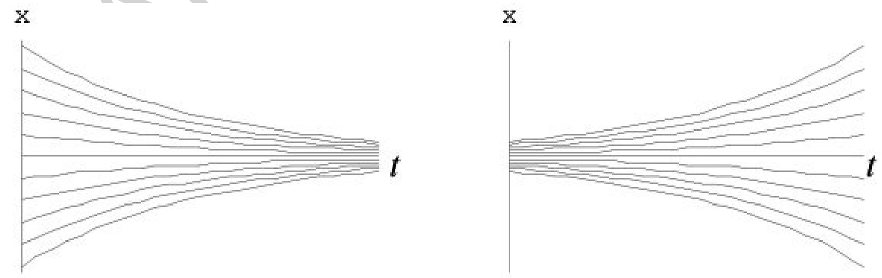
Its solution is

$$z(t) = \left(z(0) - \frac{s}{\delta} \right) e^{-(1-\alpha)\delta t} + \frac{s}{\delta}.$$

Substituting $z = x^{1-\alpha}$ into the above solution yields the final solution

$$k(t) = \left[\left(k(0)^{1-\alpha} - \frac{s}{\delta} \right) e^{-(1-\alpha)\delta t} + \frac{s}{\delta} \right]^{1/(1-\alpha)}.$$

We see that as $t \to +\infty$, $k(t) \to (s/\delta)^{1/(1-\delta)}$. We depict the model in Figure 2.



(a) $a < 0$; exponential decay (b) $a > 0$; exponential growth

Figure 2: The Solow-Swan growth model

2.2. Difference Equations

A typical linear homogenous first-order equation is given by

$$x(t+1) = a(t)x(t), \quad x(t_0) = x_0, \quad t \ge t_0 \ge 0, \tag{3}$$

where $a(t) \neq 0$. One obtains the solution of Eq. (3) by a simple iteration

$$x(t_0+1) = a(t_0)x_0,$$
$$x(t_0+2) = a(t_0+1)x(t_0+1) = a(t_0+1)a(t_0)x_0,$$
$$x(t_0+3) = a(t_0+2)x(t_0+2) = a(t_0+2)a(t_0+1)a(t_0)x_0.$$

Inductively, we obtain

$$x(t) = x(t_0+t-t_0) = \left[\prod_{i=t_0}^{t-1} a(i)\right] x_0. \tag{4}$$

Example: There are $2n$ people. Find the number of ways, denoted as $p(n)$, to group these people into pairs. To group $2n$ people into pairs, we first select a person and find that person a partner. Since the partner can be taken to be any of the other $2n-1$ persons in the original group, there are $2n-1$ ways to form this first group. We now are left with the problem of grouping the remaining $2n-2$ persons into pairs, and the number of ways of doing this is $p(n-1)$. Thus we have

$$p(n) = (2n-1)p(n-1).$$

Since two people can be paired only one way, we have $p(1) = 1$. To apply Eq. (4), we rewrite the above formula as $p(n+1) = (2n+1)p(n)$. According to Eq. (4), we have

$$p(n) = \left(\prod_{i=1}^{n-1}(2i+1)\right)p(1) = (2n-1)(2n-3)\cdots 1 = \frac{(2n)!}{2^n n!}.$$

The nonhomogeneous first-order linear equation associated with Eq. (3) is given by

$$x(t+1) = a(t)x(t) + g(t), \quad x(t_0) = x_0, \quad t \ge t_0 \ge 0. \tag{5}$$

The unique solution to Eq. (5) may be found as follows

$$x(t_0+1) = a(t_0)x_0 + g(t_0),$$
$$x(t_0+2) = a(t_0+1)x(t_0+1) + g(t_0+1) = a(t_0+1)a(t_0)x_0 + a(t_0+1)g(t_0) + g(t_0+1).$$

Inductively, the solution is

$$x(t) = \left[\prod_{i=t_0}^{t-1} a(i)\right] x_0 + \sum_{r=t_0}^{t-1}\left[\prod_{i=r+1}^{t-1} a(i)\right] g(r). \tag{6}$$

Example: Price dynamics with adaptive expectations

There are two financial assets available to investors; a riskless bank deposit yielding a constant rate r in perpetuity, and a common share, that is, an equity claim on some firm, which pays out a known stream of dividends per share, $\{d(s)\}_{s=t}^{\infty}$. Let $p(s)$ be the actual market price of a common share at the beginning of period s, before the dividend $d(s) > 0$ is paid. Suppose also that the future share prices are unknown but that all investors have the common belief at $t = s$ that the price is going to be $p^{\mathrm{e}}(s+1)$ at the beginning of the following period. We consider the following arbitrage condition: $(1+r)\,p(t) = d(t) + p^{\mathrm{e}}(t+1)$, meaning that if a monetary sum of $p(t)$ dollars were invested in the stockmarket at time t, it should yield at $t+1$ an amount whose expected value $d(t) + p^{\mathrm{e}}(t+1)$ equals the principal plus interest on an equal sun invested in bank deposits. The adaptive expectation hypothesis is described by

$$p^{\mathrm{e}}(t+1) = ap(t) + (1-a)\,p^{\mathrm{e}}(t),$$

where the parameter, $a \in [0, 1]$, describes the speed of learning. Using the arbitrage condition to eliminate expected prices from the adaptive expectation equation yields

$$p(t+1) = \lambda p(t) + b(t),$$

where

$$\lambda \equiv (1+r)\frac{1-a}{1+r-a}, \quad b(t) \equiv \frac{d(t+1) - (1-a)d(t)}{1+r-a}.$$

We have $\lambda \in (0, 1)$ if $r > 0$ and $a \in (0, 1)$. The solution of $p(t+1) = \lambda p(t) + b(t)$ is given by

$$p(t) = \lambda^t p(0) + \sum_{i=0}^{t-1} \lambda^i d(t-i).$$

Another special case of Eq. (5) is that both $a(t)$ and $g(t)$ are constant. That is

$$x(t+1) = ax(t) + b, \quad x(t_0) = x_0, \quad t \ge t_0 \ge 0. \tag{7}$$

Using formula (6), we solve Eq. (7) as

$$x(t) = \begin{cases} a^t x_0 + b\left(\dfrac{a^t - 1}{a-1}\right), & \text{if } a \ne 1, \\ x_0 + b, & \text{if } a = 1. \end{cases}$$

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Biographical Sketch

Dr. Wei-Bin Zhang, born in 1961, teaches and does research at Ritsumeikan Asia Pacific University in Japan. His main research fields are nonlinear economic dynamics, theoretical economics, growth theory, monetary economics, regional and international economics, urban economics, East Asian industrialization, and Chinese philosophy. He has published more than 100 academic articles and 19 books. His books include Synergetic Economics (1991), Confucianism and Modernization (1999), The American Civilization Portrayed in Ancient Confucianism (2003), Economic Growth Theory (2005), Differential Equations, Bifurcations, and Chaos in Economics (2005), Growth with Income and Wealth Distribution (2006), Discrete Dynamical Systems, Bifurcations and Chaos in Economics (2006).

He was brought up in China. He graduated from Department of Geography, Peking University. He finished graduate study at Department of Civil Engineering, Kyoto University in Japan. He obtained his Ph.D. in Economics at Department of Economics, University of Umeå in Sweden. He conducted research in Sweden from 1987 to 1998.