UNIVERSALITY IN CHAOS: EVOLUTION OF TURBULENCE

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Summary

Nature is nonlinear; this is the cause of a great variety of phenomena happening in our own life, from both the physiological and the physical point of view. Dynamical systems serve to describe the evolution in time of life and its existence in our universe. The importance of this kind of systems motivates us to go far beyond of their simple formulation and permits us to establish new mathematical tools for the study and analysis of their properties. This review chapter presents an overview of nonlinear dynamics and chaos concepts useful for the analysis of turbulence phenomena.

1. Introduction

The question about what is chaos can be established according to the specialized literature in the area in a threefold way: viewing the phenomenon, taking the word *chaos* literally and studying the science of chaos. In 1831 Faraday observed superficial water waves in a container vibrating vertically with the frequency ω and discovered the appearance of subharmonic components of frequency $\omega/2$.

This experiment -*the phenomenon*- has been repeated again and again to through some light on the new understanding of chaos. Lord Rayleigh repeated and discussed the experiment from the parametric resonance point of view in his treatise, *The Theory of Sound*.

The most important thing to be taken into account for this experiment is the existence of subharmonics. In physics many experimental devices are used as frequency transformers to distinguish the nature of several phenomena; this means that a system can be considered as linear if the frequency components of both the input and the output signals coincide, and nonlinear otherwise. In nonlinear systems, higher harmonics as well as sums and differences of the input frequencies appear naturally, but it is not trivial to generate subharmonics. The subharmonic modes present very small amplitudes, no matter how weak the nonlinearity could be. The subharmonic observed by Faraday and Rayleigh was a kind of threshold phenomenon: it appeared suddenly when the nonlinearity reached a certain value. The discovery of both, the threshold and the subharmonic required some nontrivial explanation. In 1981, the Faraday's experiment was repeated using novel and modern data acquisition and analyzing systems. In the 'new' experiment not only the second subharmonic, was found, but also a sequence of subharmonics each one with its own threshold were found out. This sequence turned into a noise-like output with a continuous frequency spectrum, which is termed today as chaos (CHs). 150 years after Faraday's experiment, researchers returned to study his experiment, with the intention to give some explanation of chaotic phenomena through the use of nonlinear dynamical systems (DS). In fact, in many nonlinear systems, chaotic states of motion may be attained via a finite or infinite series of numbers of sharp transitions while the final chaotic states are characterized by a number of quantities that distinguish them from the pure random state. The process of calculating these metric properties has been called characterization of CHs, a subject that will be studied in the next sections.

On the other hand, it is known that nature is essentially nonlinear and the idea that natural processes have regular behavior is a consequence of linear paradigms. The excessive use of linear analysis had limited the comprehension of natural processes for many years. One of these paradigms is the strict determinism which has delayed the recognition of chaotic phenomena, since from the deterministic point of view, it can be known at every time and every position all what is concerned with respect to positions, motions and general effects of all the entities that conform a body, as Laplace established in the XVIII century. Nevertheless, in 1892, Poincaré studied the dynamical response of the three-body problem. The analysis performed by Poincaré included another point of view that contrasted to the Laplacian determinism. Although Poincaré had an absolutely clear vision with respect to CHs as it is understood nowadays, it was only in 1963, when Lorenz developed studies about meteorology; that the idea of CHs was related to DS and was reconsidered again, through the use of modern computers with graph packages and simple algorithms. Lorenz studied the classical problem of Rayleigh-Bénard for fluid convection, which contains two parallel plates, separated by a fluid, where both plates were subjected to some temperature gradient. Lorenz's analysis shows that small variations on initial conditions may cause great changes in the system's response; a recognition of this fact caused the start of the modern study of CHs. This phenomenon, representing sensitive dependence on initial conditions, is a characteristic feature of CHs. Colloquially, this effect became famous as the *butterfly effect*, which means that if a butterfly flaps its wings in Japan, then it may cause a hurricane in México.

The word *chaos* appeared as a scientific term with L. Boltzmann, who assumed molecular CHs in his derivation of the famous H-theorem more than 100 years ago. N. Wiener used the word CHs in the titles of several papers. Both scientists, however, used it to denote disorder caused by or closely related to stochastic processes. The modern usage of the word chaos meaning intrinsic stochastic behavior in deterministic systems appeared for the first time in the Li and Yorke's paper entitled "Period Three implies chaos". Together with these pioneering studies, many relevant contributions may be found in the study respect CHs. Just as a tribute to some important authors, one might mention: Feigenbaum, Smale, Shaw, Duffin, Van der Pol, Yorke, Ott, Guckenheimer, Holmes, Grebogi, Moon, Abarbanel, Thompson, Chua, Cvitanovic, Pomeau, Parisi, Brézin, Percival, Ruelle, Mandell, Prigogine, etc.

About the same time, Mandelbrot studying DS, established the existence of the fractal geometry of nature in contrast to classical differential geometry, which provides just a first approximation to the structures of physical objects. Fractals have been observed in nature in different situations varying from geometry to physical sciences. Basically, it is possible to categorize fractals into two different groups: solid objects and strange attractors. The first type includes physical objects which exist in ordinary physical space. On the other hand, the second type considers conceptual objects embedded in the state space (see Section 2.1) of chaotic DS. At the present time, there is still no generally accepted definition for chaos.

Indeed there have been some rigorous mathematical definitions for complex behavior in DS; however, it is usually very difficult to fit a realistic system or model into some mathematical framework. Therefore, one tends to use a working operational definition for CHs. If ostensibly random motion occurs in a system, without applying any external stochastic force, and the individual output depends on the initial conditions sensitively, but, at the same time, some global characteristics (e.g. a positive Lyapunov exponent or Kolmogorov entropy, fractal attractor dimension, etc.) turn out to be quite independent of the initial conditions, then chaotic behavior appears. It can be characterized through a time series tool that will be considered in Section (4). Nowadays, the use of symbolic dynamics (see Section 2.5) provides us a rigorous way to define CHs. In order to capture the essence of a physical phenomenon, one has to set aside all the secondary factors and to construct simple, nontrivial, mathematical models that in the first instance can possibly describe them. Since few models for DS can be solved rigorously, it is often necessary to resort to some approximations; fortunately nowadays, the barrier between analytically solvable and unsolvable models has been diminishing since the introduction of high-speed digital computers. In the 1990s, the analysis of chaotic behavior became quite common but not yet the rule in many different fields of science as physics, chemistry, engineering, medicine, ecology, biology, and economy.

2. The Universality

Successes in the theory of phase transitions and critical phenomena have provided a new meaning in regards to universality. Many natural phenomena and mathematical models are grouped into certain universality classes characterized by a similar behavior in their dependence; especially, when their parameters are close to some critical value where abrupt change takes place. As a rule, a large number of degrees of freedom are required to describe the evolution of a complex physical system. The sudden change of the system at a certain transition point, may be characterized, however, only by a few variables. Generally speaking, it is the sharp nature in the transitions rather than smooth evolution of the process that reveals the universal nature of DS. However, some models are studied as representatives of relevant universality classes (see Section 2.2, and 2.3). Historically, there have been a lot of models which notably stimulated the development of science. Some of them have served as archetypes for the development of many important theories. In the very first place, one can remember the two-body problem, starting from the classical Keplerian problem of celestial motion, concerning the explanation of Mercurian perihelion precession in relativistic theory and of the hydrogen atom spectra in quantum mechanics, culminating in the impact of the understanding of the Lamb shift on the development of non-relativistic and relativistic quantum field theory. The Brownian motion is the second example, which has been the seed to construct the whole stochastic approach in the physical sciences: from Langevin to Fokker-Planck equation, path integral representation of Wiener to the Onsager-Machlup functional. In studying chaotic behavior in nonlinear systems, there exists some other of such archetypes, namely, the one dimensional mapping of the interval which will be considered in Section (2.2). Universality concepts and some generalities will be treated in the following sections.

2.1. The Dynamical Systems

A DS may be mathematically expressed either by a set of first-order differential equations, or by a system, called a map, as follows:

$$\dot{x}_1 = \frac{dx_i}{dt} = f_i(x,t), \quad i = 1,...n \quad x \in R^n. \tag{1}$$

$$X_{i+1} = F(X_i), \quad X \in R^n. \tag{2}$$

The system is called autonomous if $f_i = f_i(x)$, i.e. the functions do not change with time. Systems of higher-order differential equations can be reduced to first-order systems by a suitable change of variables. This type of formulation is general to Hamilton representations of conservative energetic systems. The complex patterns which appear in nature are vastly studied in different fields of sciences. Spatial aspects of DS are included considering coordinates in partial differential equations as mathematical models. On this basis, a DS may be understood as a transformation f that is imposed to a vector field x. A DS is generally defined on a configuration space consisting of a topological manifold, called state space or phase space. In brief, *topology* is the study of those properties that are unaltered by homeomorphisms, one-to-one continuous transformations whose inverses are also continuous and provides the tools to understand global aspects related to DS . A manifold is locally like a Euclidean space $\Re_n$, but may have varied global structures, as exemplified by the cylinder, the torus, the Klein bottle and other higher-dimensional spaces. Integration of (1) to $x_i(t) = I_i(x(0),t), \quad i = 1,...n$, yields integral curves or trajectories forming a flow on the manifold. The sets of these flow curves are called orbits. The flow thus integrates the field of velocity vectors determined by Eq. (1). Linear differential equations admit analytic solutions and have well-defined asymptotic behavior as $t \to \infty$, converging to fixed points, or periodic oscillations, forming closed orbits. An *equilibrium point* (or fixed point) is a special point of the state space where the system may stay stationary, which means that the solution does not vary with time. Therefore, if $\bar{x} \in R^n$ is an equilibrium point of the system, then $f(\bar{x}) = 0$. In the same way, for a map, $\bar{X} \in R^n$ is an equilibrium point if $\bar{X} = F(\bar{X})$.

In contrast, even the simplest deviations from linearity, including quadratic, bilinear and piecewise-linear functions can, under suitable conditions, result in a more complex chaotic behavior, in which the orbits of the system are attracted to a complex higher-dimensional subset called a strange attractor, or to be ergodic. Ergodic flows behave like thermodynamic systems in the sense that they can be modeled over statistical ensembles because the orbits fill a set of invariant measure. An attractor is a subset of the manifold to which an open subset of points, the basin of the attractor, tends towards a limit set with increasing time. Existence of an attractor requires a local volume to be contracted with increasing time and is consistent with a dissipative system (see Section 5.4), in the position-momentum representation. The highlight of a chaotic flow is a sensitive dependence on the initial conditions. Points which are arbitrarily close initially, become exponentially further apart with increasing time, leading to the amplification of very small perturbations into global uncertainties. Sensitivity results in both, an entropy increase associated with the loss of positional information with time, and in structural unsteadiness in which an arbitrarily small perturbation of the flow causes structural changes to the topology of the orbits although they may have similar qualitative behavior. The entropy results in a loss of memory of the initial conditions in any numerical approximation over time. Time dependent systems are capable of abrupt changes in their topological form called *bifurcations* (see Section 3.4), as the underlying parameters cross critical values. Bifurcations results in abrupt disastrous changes in the topology of the flow under continuous variation of the time dependent parameters. Because nonlinear differential equations cannot in general be integrated directly, it is often necessary to resort to techniques of numerical integration in which a discrete transfer function is constructed, which approximates a stroboscopic representation of the flow, at discrete time intervals

$$x(k\Delta t) = G^k(x(0)), \quad k = 1, 2, 3, \ldots, \tag{3}$$

by using numerical methods such as the improved Euler method or Runge-Kutta. Hence the following expansion can be obtained:

$$x_i(t+\Delta t) = G_i(x(t)) \approx x_i(t) + \frac{\Delta t}{2}\left[F_i(x(t)) + F_i(x(t) + \Delta t F_i(x(t))\right]. \tag{4}$$

In a time varying system, CHs may become established by three principal routes involving a (possibly infinite) sequence of bifurcations of the attractor, intermittent disruption of a periodicity, or the topological breakup of a surface, such as a torus, representing several linked oscillations. These routes are studied, because a knowledge of all of them is essential, for instance, in characterizing chaotic dynamics of a dissipative system as in turbulence, or in biomedical systems; as is the case for the brain or in excitable cells (e.g. heart cells behavior).

2.2. The Logistic Map

Many researchers have dedicated their efforts to analyze CHs in different fields of science. Robert May, for example dealt with a system related to the insect population dynamics. His work became known as the *logistic map*; he evaluated the insect population in one year, X_{i+1}, from the previous year, X_i.

$$X_{i+1} = \eta X_i (X_i - 1). \tag{5}$$

The parameter η defines environmental characteristics. There is no doubt about the simplicity of this mathematical model, however, the diversity of its dynamical response is very rich. The linear differential equation (5) can be easily solved: $X_{i+1} = X_0 \eta^i$, which states that, if, on average, each insect lays η eggs and all eggs hatch, then the population will grow exponentially, provided $\eta > 1$. Taking into account for the interactions between insects which fight and kill each other for limited food, contagious epidemic, etc., Eq. (5) is modified into

$$X_{i+1} = \eta_1 X_i - \eta_2 X_i^2. \tag{6}$$

Despite its apparently simple form, Eq. (6) exhibits a complex dynamical behavior. One of the two parameters η_1, or η_2, can be scaled out, normalizing X_{i+1} and writing it down in a form which appears more frequently in the literature,

$$X_{i+1} = 1 - \mu X_i^2, \quad X_i \in (-1,1), \;\; \mu \in (0,2). \tag{7}$$

Using a computer, covering the parameter range $\mu \in (0,2)$ and recursively computing in small steps of time for each parameter value μ, one can arrive at the bifurcation diagram in the $X-\mu$ plane (see Figure 1).

For $\mu \leq 0.75$, the limiting set consists of one point. This is the *fixed point* of the mapping. For $\mu = 0.75$ to 1.2 the limiting set comprises two points, giving rise to a 2-cycles or period 2 orbits. Then thereafter one obtains consecutively the $4, 8, \ldots 2^i$ ...cycles, forming a period-doubling bifurcation sequence (see Figure 1). Downwards along the μ axis, it can be observed a series of 2^i bands of chaotic behavior. A bifurcation diagram is essentially a diagram of attractors by its behavior to be attracted, generating a sufficient number of transients. Fixed points and periodic points are trivial attractors, while the darkened vertical segments are chaotic attractors.

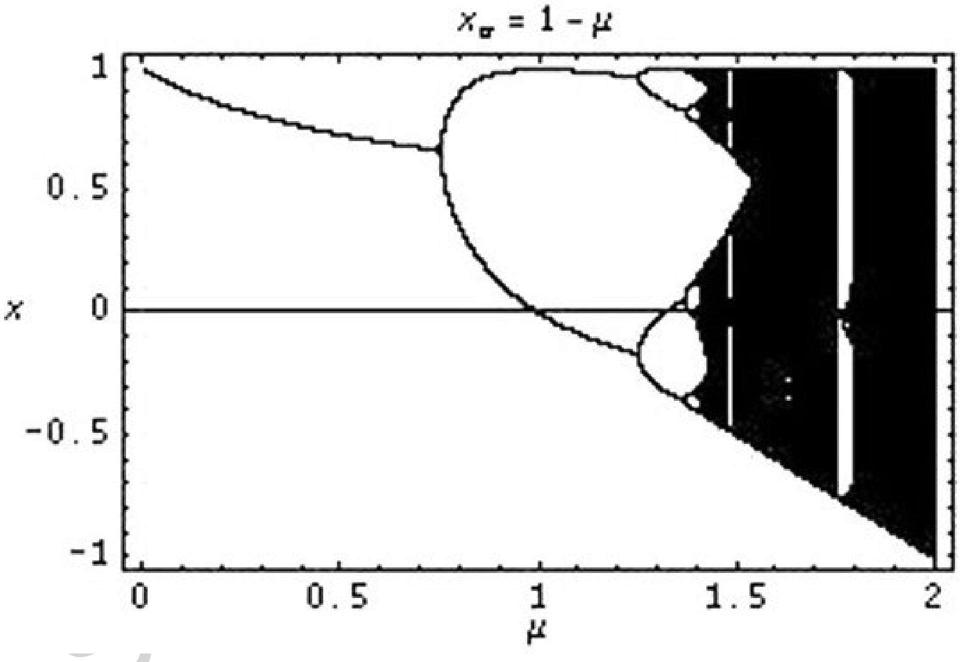

Figure 1. The bifurcation diagram for the logistic map

A self-similar structure can be seen. It is remarkable, that all these features happen to be shared by many other nonlinear systems. Both, the global structure of the bifurcation diagram and the numerical characteristics of many local transitions within the diagram are universal properties characterized by two parameters denominated δ and α and discovered by Feigenbaum in 1975, these were derived from a mathematical model of animal population, studying the fixed points of the logistic map, and characterizing the geometrical approach of the bifurcation parameter to its limit value when the parameter μ is increased for fixed x. The δ constant is calculated using the period-doubling region, and evaluating the distance between successive bifurcation points, μ_n, which shrinks geometrically in such a fashion that the ratio of the intervals defines it.

$$\delta = \lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}} = 4.669201609102990... \tag{8}$$

The α parameter or associated *reduction parameter* is defined as the separation of adjacent elements of period doubled attractors (see figure 2)

from one double to the next, has a value of

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = -2.502907875..., \tag{9}$$

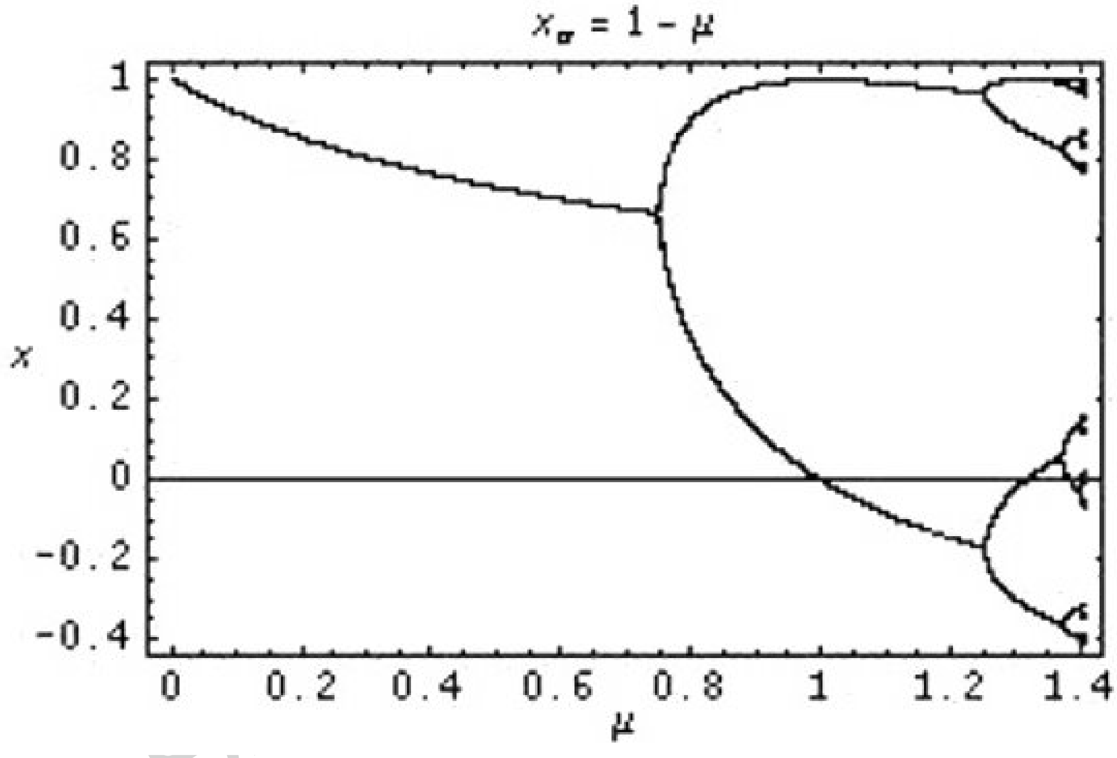

Figure 2. The successive bifurcations used to evaluates the α parameter

where d_n is the value of the nearest cycle element to 0 in the 2^n cycle. Until 1999, both quantities have been evaluated to 1018 decimal places. Amazingly, the Feigenbaum constants δ and α are universal for all one-dimensional maps $f(x)$, if the map has a single locally quadratic maximum. This was conjectured by Feigenbaum, and demonstrated rigorously by Lanford in 1982 for the case $r = 2$, and by Epstein in 1985, for all $r < 14$ in the map $f(x) = 1 - \mu|x|^r$ with r an generic test exponent. The most significant contribution of M. J. Feigenbaum comprises not only the discovery of these universal constants, but also his device of a universal, renormalization group approach, which reveals the physical meaning of these constants and opens up a way to calculate them with high precision, independently of any particular model. This characteristic feature permit us make to some predictions concerning some other problem under study but pertaining to the same universality class. The behavior of this quadratic map is typical for many DS. One year after their discovery, the period-doubling route to CHs and the constants δ and α, appeared in some equations used to describe hydrodynamic flow, which will be treated in Section (5). The logistic map serves as a representative example of a wide class of one dimensional mapping, i.e., of functions with only one hump. Those universal properties that depend on only one local maximum, but not on the nature of the maximum, are sometimes called of *structural universality*. Close to the local maximum at $X = X_0$, and if the nonlinear functions f may be expanded in the form

$$h(X) = h_{\max} - \eta(X - X_0)^{\beta} + ..., \tag{10}$$

those properties, which are shared by maps with one and the same value of β, are classified as belonging to some kind of *metric universality*.

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Biographical Sketches

Juan José Godina Nava, was born at Durango city, México. He is faculty member in Physics Department at Research Center and Advanced Studies of National Polytechnic Institute (CINVESTAV IPN) as a Researcher. Is member of SNI level I. He obtained his PhD degree in 1994 at Physics Department in CINVESTAV IPN. From 1997 to 1999 performed a postdoctoral training at Iowa University USA, advised by Professor Yannick Meurice in Lattice Field Theory. Their areas of research are: Phenomenology in particles physics, Quantum Field Theory, Medical Physics and Dynamical Systems. He has been published 28 articles in their areas of research and published one specialized book. He has participated in several international conferences and seminars, and International Workshops and Symposiums with invited talks and different works similarly in Congress and National meeting. He is a member of the Mexican Society of Physics, honorary member of the Society of Science, Culture and Arts A.C. ISSCultArt A.C. He has advised 7 Bachelor degrees, 4 Master degrees and 4 PhD degrees.

Miguel Angel Rodríguez Segura was born in Federal District, México. He collaborates actually in the Physics Department at Research Center and Advanced Studies of National Polytechnic Institute (CINVESTAV IPN) as a Researcher. He obtained his PhD degree in 2005, in mathematical physics at Physics Department in CINVESTAV IPN. Their areas of research are: Mathematical Physics, Medical Physics and Dynamical Systems. He has been published 8 articles in their areas of research. He has participated in several international conferences and seminars, and International Workshops and Symposiums. He has advised 2 Bachelor degrees.

Guillermo Arnulfo Vazquez Coutiño, was born in Chiapas, Mexico. He obtained his B.Sc. at the National Autonomous University of Mexico in chemical engineering in 1985. He worked as heat exchanger and fluid flow design engineer for the Mexican Institute of Petroleum from 1980 to 1985, where he specialized in the design of optimization of heat exchanger networks for the petroleum industry and developed a computer program for the design of air coolers. In 1990 he obtained his M.Sc. in chemistry at the Metropolitan Autonomous University Campus Iztapalapa, where he presently works as professor and researcher. He is doing his Ph.D. in Theoretical Physics at the Metropolitan Autonomous University. His research areas are complex systems theory, non linear dynamical systems, applied fractional calculus, theoretical electro chemistry and applied category theory. He is also the inventor of the language of chemical operands, of the theory of ball theory of quality control, and he has been in charge of the development of what he called Recurrent Algorithmic Spectroscopy RAS, which has found applications in different types of spectroscopy's , like UV,IR,MNR, electrochemical and spectra chemical information, among others research themes. He has published 5 articles related to Chaos Theory and 5 more in mathematical areas, chemistry and mathematical physics. He has participated in several international conferences and seminars, and International Workshops and Symposiums as speaker of invited talks. He has participated as organizer of several international physical and mathematical events. He has advised 16 Bachelor degree thesis and 7 Master degree thesis. He is a member of the Mexican

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Gregorio Serrano Luna, was born in Mexico City. He obtained his B.Sc. at National Polytechnic Institute, in Physics and Mathematics, 1986, in 1992 obtained his M.Sc. at CINVESTAV IPN, in Bio-electronics, where actually work as Assistant-research. Their research areas are biomedical instrumentation, complex systems, and dynamical systems. He has been published 5 articles related to ELF Magnetic Field Therapy, and 4 more in theoretical and physiological topics in ELF magnetic field. He has participated in several international conferences and seminars, and International Workshops and Symposiums.