NONLINEAR ANALYSIS OF FRAME STRUCTURES

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Contents

Summary

Nonlinear methods of analysis for seismic design of buildings and bridges typically rely on nonlinear frame analysis techniques. Recent advances in modeling capabilities have made these analyses applicable for both researchers and practicing engineers. Commercial and research programs model structural members using either distributed or lumped plasticity. In this chapter, emphasis is given to the first approach that has reached a high level of precision thanks to advanced formulations. Details are given for the element formulations and their implementation in a computer code. Both material and geometric nonlinearities are presented. Numerical integration and localization

issues are discussed too. Special elements for regions where beam theory does not apply and for elements with bond-slip are also briefly discussed.

1. General

A frame structure is a three-dimensional solid made mainly of beams and columns. In the general case, a three-dimensional model should be built. In order to reduce the complexity of the notation and to simplify the figures, only two-dimensional frames are considered here. In most cases, the extension to the space problem is straightforward. For buildings, Figure 1 shows the most common structural members. According to Schlaich et al. (1987), a frame is made of B regions, that is regions where the classical Beam theory applies, and of D regions, or Discontinuities where the beam theory does not apply. Typical B regions are slender beams and columns, D regions are the beam-column joints, corbels and squat members in general. This chapter focuses on the nonlinear modeling of the B-regions.

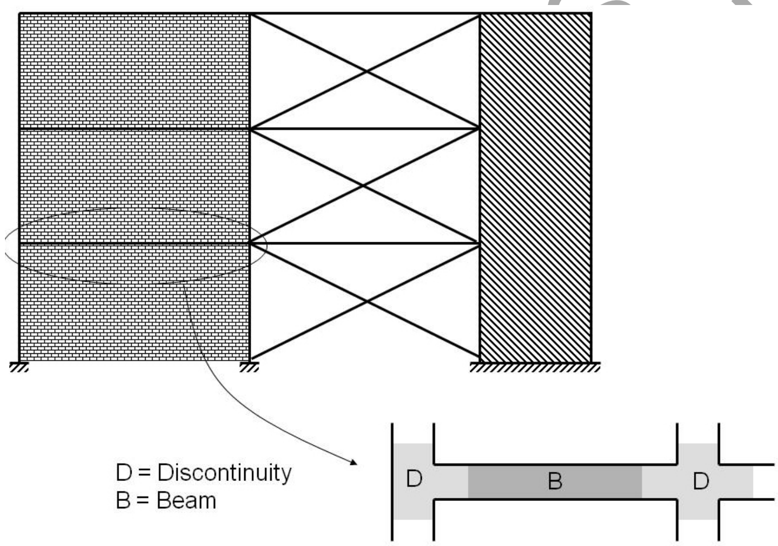

Figure 1. A frame and its main components

There exist two sources of nonlinearities: material and geometric. Material nonlinearities are treated through the constitutive laws of the materials used, while nonlinear geometry becomes important when large deformations and/or large displacements take place.

In the first part of this chapter, linear geometry is assumed: the deformations are assumed small and equilibrium is imposed in the undeformed configuration. At the end of this chapter, nonlinear geometric effects are treated in a dedicated section. The differential equations governing the main beam problems are briefly reviewed hereafter.

1.1. Differential Equation of Axial Problem

Given an infinitesimal frame length *dx* (Figure 2), in the axial problem only axial forces and deformations are considered. Plane sections are assumed to remained plane here. Combining the following equilibrium, compatibility and constitutive laws

$$\frac{d}{dx}\left[N(x)\right] = -w_x(x) \tag{1}$$

$$\varepsilon_0 = \frac{du_0}{dx} \tag{2}$$

$$N(x) = EA(x)\ \varepsilon_0 \tag{3}$$

the governing differential equation is

$$\frac{d}{dx}\left[EA(x)\ \frac{du_0}{dx}\right] = -w_x(x) \tag{4}$$

where $N(x)$ is the axial load, $E(x)$ is the material modulus of elasticity (assumed here uniform over the section but variable along the beam length), $A(x)$ is the cross section, which may vary along the beam, $u_0(x)$ and $\varepsilon_0(x)$ are the axial displacement and the axial strain, respectively, at the section reference axis along the beam and $w_x(x)$ is the distributed axial load in the axial direction. $EA(x)$ is the section axial stiffness. Appropriate essential and natural boundary conditions are needed to complete the differential equation.

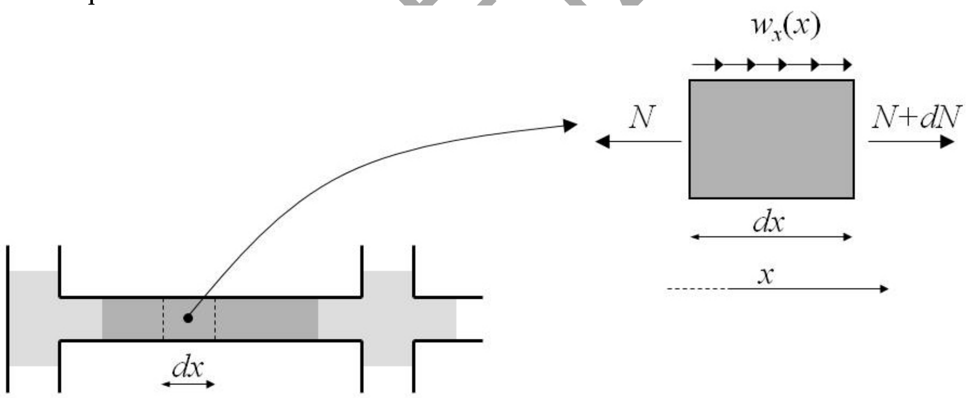

Figure 2. Infinitesimal beam length: axial problem

In the case of constant EA and no axial load

$$EA\frac{d^2u_0}{dx^2} = 0 \tag{5}$$

The exact solution in this case is

$$u_0(x) = c_0 + c_1 x \tag{6}$$

where c_0 and c_1 are constants that are computed from the boundary conditions.

1.2. Differential Equation of Torsion Problem

Warping is neglected in the derivation of the differential equation for the torsion problem. The infinitesimal beam length is shown in Figure 3. If T is the torque, $\theta(x)$ the angle of twist, $GJ(x)$ is the torsional stiffness, and $m_x(x)$ is the torque per unit length applied to the beam, the torsion problem is formally identical to the axial problem.

The governing differential equation is

$$\frac{d}{dx}\left[GJ(x)\frac{d\theta}{dx}\right] = -m_x(x) \tag{7}$$

For GJ constant

$$GJ\frac{d^2\theta}{dx^2} = 0 \tag{8}$$

Thus the exact solution θ is a linear function of x similar to that of Eq. (6).

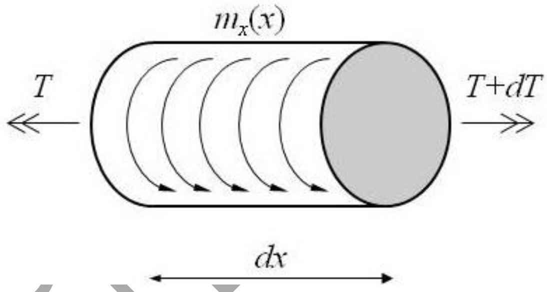

Figure 3. Infinitesimal beam length: torsion problem

1.3. Differential Equation of Bending Problem

Two main beam theories are recalled here: the Euler-Bernoulli beam theory, that considers only bending deformations, and the Timoshenko theory that considers both bending and shear deformations.

1.3.1. Euler-Bernoulli Beam Theory

The fundamental assumption of the Euler-Bernoulli beam theory is that *plane sections remain plane and normal to the beam longitudinal axis*. This is shown in Figure 4.

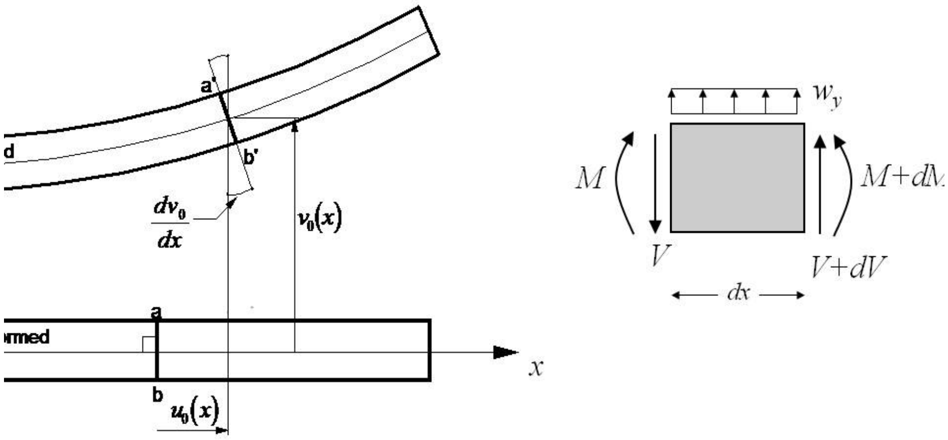

Figure 4. Deformation of Euler-Bernoulli beam and forces on beam infinitesimal length

Combining the following equilibrium, compatibility and constitutive laws

$$\frac{d^2}{dx^2}\left[M(x)\right] = w_y(x) \tag{9}$$

$$\kappa = \frac{d^2 v_0}{dx^2} \tag{10}$$

$$M(x) = EI(x)\,\frac{d^2 v_0}{dx^2} \tag{11}$$

the governing differential equation is

$$\frac{d^2}{dx^2}\left[EI(x)\,\frac{d^2 v_0}{dx^2}\right] = w_y(x) \tag{12}$$

In the previous equations $M(x)$ is the bending moment, $I(x)$ is the cross section second moment of inertia, which can vary along the beam, $v_0(x)$ and $\kappa(x)$ are the vertical displacement and the curvature, respectively, at the section reference axis along the beam, $w_y(x)$ is the distributed vertical load and $EI(x)$ is the section bending stiffness. If $EI = const$, then

$$EI\,\frac{d^4 v_0}{dx^4} = w_y(x) \tag{13}$$

In the absence of distributed load:

$$EI\,\frac{d^4 v_0}{dx^4} = 0 \tag{14}$$

Thus, the differential equation of the Euler-Bernoulli beam is of the 4th order in the unknown displacement $v_0(x)$.

Four boundary conditions (essential or natural) must be added to the above differential equations in order to find the solution to a given problem. At least two of these must be essential, otherwise there is a mechanism.

1.3.2. Timoshenko Beam Theory

The Timoshenko beam theory represents a simplification of more precise beam theories that account for shear deformations.

The fundamental assumption of the Timoshenko beam theory is that *plane sections remain plane, but no longer perpendicular to the beam axis due to the shear deformation*. Figure 5 shows the implications of such assumption. The cross section remains plane (contrarily to what happens according to Jourawsky's theory) and the shear deformation γ is constant over the cross section.

The vertical displacement of the beam reference axis is the sum of the flexural and shear deflections, as shown in Figure 6:

$$v_0(x) = v_f(x) + v_s(x) \tag{15}$$

where v_f is the deflection due to flexure only, and v_s is the deflection due to shear only.

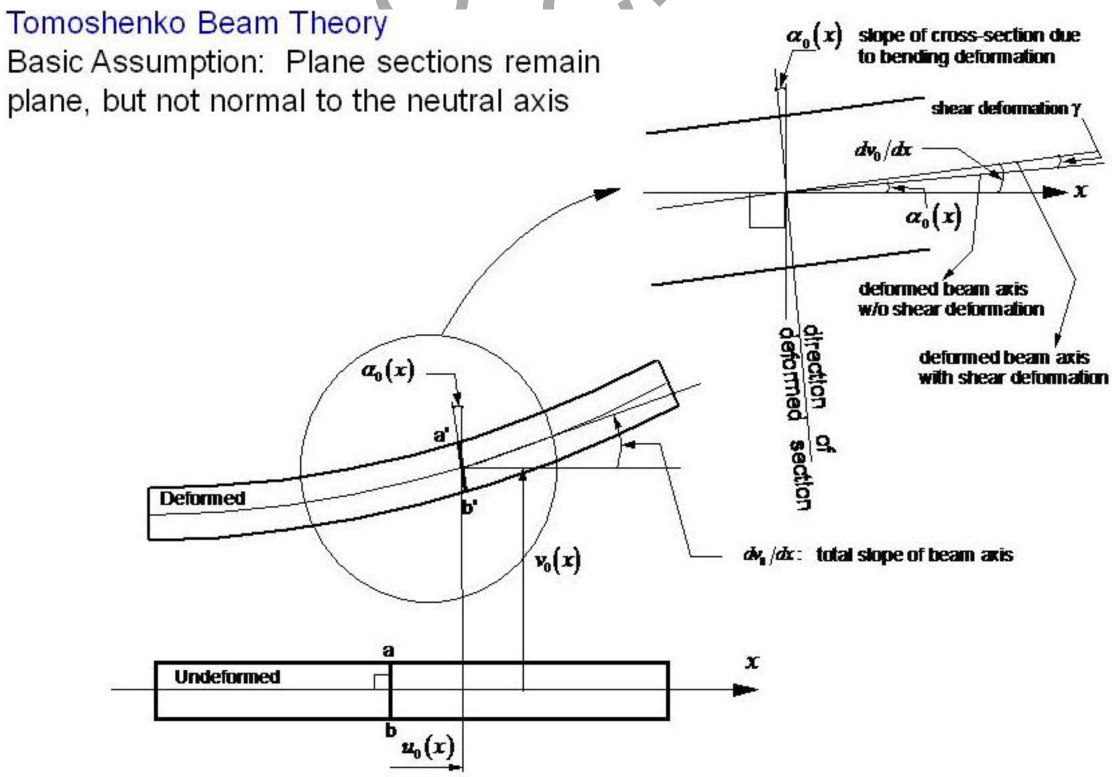

Figure 5. Deformations of Timoshenko beam

It follows that

$$\frac{dv_0}{dx} = \underbrace{\frac{dv_\text{f}}{dx}}_{\alpha_0} + \underbrace{\frac{dv_\text{s}}{dx}}_{\gamma} \tag{16}$$

which implies that the total slope of the beam axis, dv_0/dx, is the sum of the rotation due to bending ($\alpha_0 = \frac{dv_\text{f}}{dx}$) and of the rotation due to shear ($\gamma = \frac{dv_\text{s}}{dx}$).

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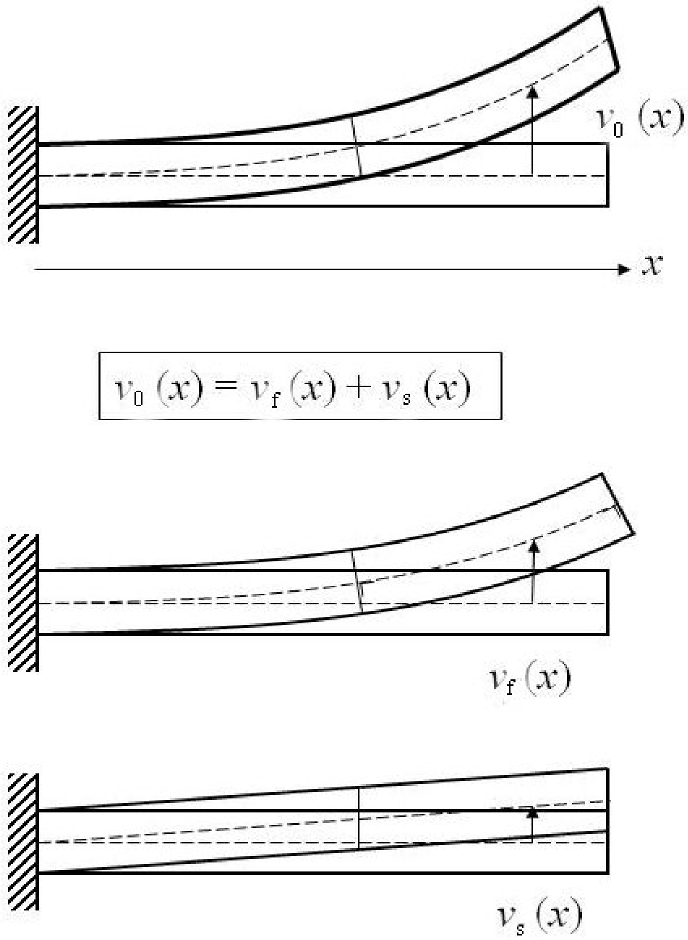

Figure 6. Displacements of Timoshenko beam

In the deformed configuration the beam cross section remains plane but rotates by an angle $\alpha_0(x)$, which is different from the rotation of the beam axis ($dv_0/dx = \alpha_0 + \gamma$) due to the effect of shear deformation. The Euler-Bernoulli beam theory neglects the effect of shear deformation (i.e., $\gamma = 0$), and therefore assumes that the beam axis and the beam cross section rotate by the same amount.

Because the problem unknowns are two, two differential equations are needed. The following two equilibrium statements are used

$$\begin{aligned} &\frac{d}{dx}\left[V(x)\right] + w_y(x) = 0 \\ &V(x) + \frac{dM(x)}{dx} = 0 \end{aligned} \tag{17}$$

and they are combined with the compatibility statements

$$\begin{aligned} &\gamma = \frac{dv_0}{dx} - \alpha_0 \\ &\kappa = \frac{d\alpha_0}{dx} \end{aligned} \tag{18}$$

and the constitutive laws

$$\begin{aligned} &V = GA_s(x)\,\gamma \\ &M = EI(x)\kappa \end{aligned} \tag{19}$$

where A_s is the shear area , typically expressed as $A_s = kA$, where $k = \frac{5}{6}$ for rectangular cross sections and $k = \frac{9}{10}$ for circular cross sections.

Combining the above relations, the two following general differential equations are obtained:

$$\begin{aligned} &\frac{d}{dx}\left(GA_s\left(\frac{dv_0}{dx} - \alpha_0\right)\right) + w_y(x) = 0 \\ &GA_s\left(\frac{dv_0}{dx} - \alpha_0\right) + \frac{d}{dx}\left(EI\,\frac{d\alpha_0}{dx}\right) = 0 \end{aligned} \tag{20}$$

If $EI = const.$ and $GA_s = const.$ (uniform prismatic beam)

$$
\begin{aligned}
&GA_s\left(\frac{d^2v_0}{dx^2}-\frac{d\alpha_0}{dx}\right)+w_y\left(x\right)=0\\
&GA_s\left(\frac{dv_0}{dx}-\alpha_0\right)+EI\,\frac{d^2\alpha_0}{dx^2}=0
\end{aligned}
\qquad (21)
$$

These are two coupled 2nd order differential equations in the unknown functions $v_0(x)$ and $\alpha_0(x)$, and they govern the Timoshenko beam problem. Four boundary conditions (essential and/or natural) must be added to the above differential equations in order to find the solution to a given problem.

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Biographical Sketches

Enrico Spacone completed his undergraduate studies in civil engineering at the University "La Sapienza", Roma Italy in 1987. After working for two years at the EPFL (Swiss Institute of Technology, Lausanne), he moved to the University of California, Berkeley, USA, where he received his M.S. degree (1990) and Ph.D. (1994) in Civil Engineering. In 1995 he joined the University of Colorado as assistant and then associate professor. He is currently a professor of structural engineering at the University "G. D'Annunzio" of Chieti-Pescara, Italy, where he teaches Structural Design, Structural Analysis and Seismic Design of Structures. He is also member of the teaching body of the Rose School (Centre for Post-Graduate Training and Research in Earthquake Engineering and Engineering Seismology) in Pavia, Italy, where he teaches Structural Analysis classes. In 2002/3 he was a Fulbright scholar at the University of Ljubljana, Slovenia. He has published several papers on nonlinear frame analysis, structural design, structural strengthening of concrete structures. He is also involved in research on the seismic risk reduction of the cultural heritage.

Joel Conte received his Civil Engineering Diploma (1985) from the EPFL (Swiss Federal Institute of Technology, Lausanne, Switzerland), his M.S. degree (1986) and Ph.D. (1990) in Civil Engineering at the University of California, Berkeley. He is currently Professor of Structural Engineering at the University of California, San Diego (UCSD). Prior to joining UCSD in 2001, he held faculty positions at Rice University in Houston, Texas (1990-1997) and at the University of California, Los Angeles (UCLA) (1998-2001). He teaches courses in linear and nonlinear, static and dynamic analysis of structures, random vibrations, structural reliability and risk analysis. He is also a member of the teaching body at the Rose School (Centre for Post-Graduate Training and Research in Earthquake Engineering and Engineering Seismology) in Pavia, Italy, where he teaches Seismic Reliability Analysis and Structural Analysis. In 2006/7, he was Fulbright Scholar at the University "G. D'Annunzio" of Chieti-Pescara, Italy. His research publications are in the areas of nonlinear structural modeling and analysis, earthquake engineering, probabilistic analysis and design of structures, random vibrations, shaking table dynamics, system identification and structural health monitoring.