INTRODUCTION TO CONTINUUM MECHANICS

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Summary

This chapter situates the different analyses involved in the framework of continuum mechanics together with the basic procedure that someone working in this broad area must follow to postulate and solve a problem. Through the early basic thinking skills in formulating, analyzing and solving a one dimensional problem the reasons that have motivated both the list of chapters and the sequence used in the organization of chapters of this encyclopedia volume are encapsulated. We focus on classical mechanics to introduce the importance of this subject.

1. Introduction

The aim of this chapter is to give an overview of this volume on continuum mechanics in a simplified setting. The idea is to show in a few pages the main arguments behind the selection of chapters and topics inside each of the chapters that can be found in this theme. Nevertheless, we also introduce some advanced concepts that give a state of the art in some areas of continuum mechanics to further emphasize that is a fundamental science in different analyses.

Continuum Mechanics is the branch of Mechanics that studies deformable materials (solids and fluids). This branch of knowledge is used in many engineering and scientific applications. Hence, continuum mechanics of solids and fluids is a fundamental discipline for students of civil, industrial and mechanical engineering as well as students of physics and mathematics. New interesting applications of continuum mechanics in biology are gaining a lot of attention lately. Furthermore, knowledge of continuum mechanics is fundamental to understand the biomechanics of living organism and biomolecules. The mathematical issues of continuum mechanics are an important source of inspiration and motivation for different areas of mathematics such as calculus of variations, differential geometry, the theory of dynamical systems and partial differential equations.

The aim of this encyclopedia volume is to provide a modular introduction to the subject. We have tried to cover the basic information, a list of important applications and some advanced topics. Along these lines, we hope that this volume be useful for researchers in the area as well as for beginners in the subject who are trying to approach the topic. For the latter, this volume devotes the theme level writing, that has been written as an initial reading. That leading chapter of this volume is just a broad overview of the subject that focuses on ideas without paying attention to sophisticated mathematical machinery. Nevertheless, it is a brief review that also may capture the attention of a reader familiar with the subject.

The first chapter, this one, locates the different chapters of this theme inside the framework of continuum mechanics. The analysis situates the different topics together with the steps that someone working in this broad area must follow to postulate and solve a problem. Through the basic thinking in formulating, analyzing and solving a one dimensional problem the reasons that have motivated the list of chapters as well as the sequence used in the distribution of chapters are encapsulated.

To underline the cultural values of continuum mechanics as a central subject in the field of Mechanics we have introduced the chapter: *History of Continuum Mechanics*.

The rest of the chapters not only give a state of the art of the subject but also provide motivation and hints for further study.

2. The Continuum Setting: Kinematics and Balance Equations.

Continuum mechanics analyzes material bodies describing the motion and deformation of matter through an observer in the classical Newtonian space-time framework. The chapters *Kinematics, Balance laws* , and *Thermodynamics* are devoted to the mathematical setting of continuum analysis. The study that deals with the geometry of the body in time is called kinematics. Whence, kinematics considers deformation and motion in a four dimensional setting that includes space and time. The scope of *Kinematics* is to provide a detailed introduction to this state of affairs. The chapters *Balance laws* and *Thermodynamics* are devoted to the propositions obeyed by the thermo-mechanical theory of continuum bodies. The results are known as balance laws. Here we review the basic steps that have to be followed to derive the balance equations for a continuum body. This is a preliminary step needed to introduce the rest of the analyses. We introduce the analysis using a minimal framework of complexity by examining a bar that can only stretch in one-dimension. In this case the surface of all cross-sections of the bar is A in the reference configuration and a in the current configuration. We need a coordinate $x \in [0, L]$ to locate a given cross-section in the reference configuration. Therefore, the motion is described by an equation of the form

$$y = y(x,t), \tag{2.1}$$

where $y \in [0,l]$ determines the position of the cross-section x in the current configuration (at a given time t).

The geometry and the kinematics of this problem is described by two simple quantities: the stretch $\lambda = \partial y / \partial x$ and the velocity $\dot{y} = \partial y / \partial t$. We require $\lambda > 0$ since the bar length cannot be reduced to zero.
Let ρ denote the mass density of the bar in the actual configuration and ρ_0 the mass density of the bar in the reference configuration. If there are no mechanisms that create mass then for any arbitrary subset of the whole domain, i.e. for $x_1, x_2 \in [0, L]$ it follows that

$$\int_{x_1}^{x_2} \rho_0 dx = \int_{y_1}^{y_2} \rho dy, \tag{2.2}$$

where $y_1 = y(x_1, t)$ and $y_2 = y(x_2, t)$. Since $dy = \lambda dx$ it is possible to write

$$\int_{x_1}^{x_2} \rho\lambda dx = \int_{x_1}^{x_2} \rho_0 dx,$$

from which we obtain the local (Lagrangian) form of the conservation of mass

$$\lambda\rho = \rho_0. \tag{2.3}$$

This is the first basic conservation law or balance law of continuum mechanics.

It is clear that $\dot{y} = \dot{y}(x,t)$ may be written considering the inverse function $x = x(y,t)$ as $v = v(y,t)$ where a different symbol for the velocity is used to express that it is given as a function of y. The description in terms of the reference coordinate x is denoted as the Lagrangian description (or material description), whereas the description in terms of the actual coordinate y is named the Eulerian description (or spatial description). Since acceleration is the time derivative of the velocity, it can be given as $\ddot{y} = \partial^2 y / \partial t^2$. On the other hand, if the velocity is given in terms of v, the time derivative implies $dv / dt = \partial v / \partial t + v \partial v / \partial y$. The derivative d / dt is named the *material time derivative*.

To derive the remaining balance equations for the problem at hand we need to extend to a body, i.e. to a continuum framework, the celebrated Newton's equation of dynamics

$$\frac{d(m\mathbf{v})}{dt} = \mathbf{F} \tag{2.4}$$

which states that the rate of change of the linear momentum $d(m\mathbf{v})/dt$, (linear momentum is $m\mathbf{v}$ where m is mass and $\mathbf{v}$ is velocity) equals to the resultant force on it, $\mathbf{F}$. Eq. (2.4) and the corresponding work-energy theorem are the basic building blocks to develop a complete theory of mechanical behavior.

To this end we start postulating the parallel Newton's law into the continuum framework for the problem at hand. It is known as the balance of linear momentum (First law of Euler). One can say that for any segment $[x_1, x_2] \in [0, L]$ it follows that

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho_0 \dot{y} dx = \int_{x_1}^{x_2} \rho_0 f dx + s\Big|_{x_1}^{x_2}, \tag{2.5}$$

where ρ_0 is the mass density per unit reference volume, $\rho_0\ f$ is the body force per unit reference volume and s is stress (First Piola Kirchoff stress) per unit area of the reference configuration (one can think that each term is multiplied by the area A to fully follow the notation).

Since this has been established in the reference configuration and x_1 and x_2 do not depend on time it is possible to rewrite Eq. (2.5) as

$$\int_{x_1}^{x_2} \rho_0 \ddot{y} dx = \int_{x_1}^{x_2} \left(\rho_0 f + \frac{\partial s}{\partial x} \right) dx$$

Furthermore, the latter must be obeyed for arbitrary subsets of the whole domain $[x_1, x_2] \in [0, L]$. Whence, one can finally write

$$\rho_0 \ddot{y} = \rho_0 f + \frac{\partial s}{\partial x}. \tag{2.6}$$

which is the local form of the conservation law (Eq. (2.5)). The local form is collected as a partial differential equation.

Using $dx = \lambda^{-1} dy$ in (2.5) it yields

$$\int_{y_1}^{y_2} \rho_0 \lambda^{-1} \frac{dv}{dt} dy = \int_{y_1}^{y_2} \left(\rho_0 \lambda^{-1} f + \frac{\partial s}{\partial x} \lambda^{-1} \right) dy,$$

from which the local form of this balance law in the current configuration can be written as

$$\rho \ddot{y} = \rho f + \frac{\partial \sigma}{\partial y}. \tag{2.7}$$

In Eq. (2.7) ρ is the mass density per unit deformed volume, ρf is the body force per unit deformed volume and σ is the Cauchy stress which is stress per unit area of the deformed configuration.

Eqs. (2.6) or (2.7), as opposed to Newton's equation of linear momentum, include two forces: the force ρf which is a body force (as for example gravity) and the force that is given in terms of stress and is understood to arise from internal interactions inside the body. This argument has to be reflected in the energy analysis of the body. The energy, E, for an arbitrary subset of the domain of the bar is

$$E = \int_{x_1}^{x_2} \left(\varepsilon + \frac{1}{2} \rho_0 \dot{y}^2 \right) dx, \tag{2.8}$$

where we have taken into account, as in particle mechanics or rigid body mechanics, the kinetic energy, and one additional term that is the internal energy (per unit reference volume) ε. Moreover, the power P of the forces acting on the arbitrary domain of the reference configuration is given by

$$P = \int_{x_1}^{x_2} \rho_0 f\dot{y} dx + s \left|\dot{y}\right|_{x_1}^{x_2} = \int_{x_1}^{x_2} \left[\rho_0 f\dot{y} + \frac{\partial}{\partial x}(s\dot{y}) \right] dx.$$

If we consider additionally heat transfer phenomena and let Q denote the heat that is supplied to the system one can write that

$$Q = \int_{x_1}^{x_2} \left(\rho_0 r + \frac{\partial q}{\partial x} \right) dx,$$

where q is the (material) heat flux and r is the density of radiation. The First law of thermodynamics states formally that energy cannot be created or destroyed. Mathematically this is written as

$$\frac{dE}{dt} = P + Q. \tag{2.9}$$

In the reference configuration it is possible to write Eq. (2.9) as

$$\dot{\varepsilon} + \rho_0 \dot{y}\ddot{y} = \rho_0 f\dot{y} + \frac{\partial}{\partial x}(s\dot{y}) + \rho_0 r + \frac{\partial q}{\partial x}.$$

Using Eq. (2.6) in the above equation, we obtain the local version of the First Law of thermodynamics as

$$\dot{\varepsilon} = s\frac{\partial \dot{y}}{\partial x} + \rho_0 r + \frac{\partial q}{\partial x}. \qquad (2.10)$$

which is known as the equation of energy. Eqs. (2.3), (2.6) and (2.10) are the local forms of the basic equations of continuum mechanics. Indeed, in a three-dimensional setting (and we refer for these issues to *Balance laws*) we have to consider not only balance of linear momentum but also balance of angular momentum which involves several important technical details.

We shall analyze the impact of thermodynamics in continuum mechanics by considering a *quantitative* version of the second law of thermodynamics. This provides additional equations that have to be obeyed by the constitutive quantities. There are several ways to do this and for a detailed discussion we refer to *Thermodynamics*.

In continuum mechanics, although with certain controversy, it is now a standard procedure to work with the *Clausius-Duhem inequality*. This inequality introduces the concept of entropy. In the reference configuration, one defines the entropy per unit reference volume, η, and postulates that

$$\frac{d}{dt}\int_{x_1}^{x_2}\eta dx \geq \int_{x_1}^{x_2}\frac{r}{\theta}dx + \left|\frac{q}{\theta}\right|_{x_1}^{x_2},$$

where $\theta(x,t)$ is the temperature field. Whence, entropy is defined as a quantity whose rate of change in time is related to the capacity of the body to absorb heat. Heat is related to the velocity fluctuations of atoms. It is in this sense that entropy is a measurement of disorder inside the body. The local form of the *Clausius-Duhem inequality* is easily obtained and can be written as

$$\dot{\eta} \geq \frac{1}{\theta}\left(r + \frac{\partial q}{\partial x}\right) - q\frac{\partial \theta}{\partial x}\theta^{-2}. \qquad (2.11)$$

Using Eqs. (2.10) and (2.11) one can write

$$\dot{\varepsilon} - \theta\dot{\eta} \leq s\frac{\partial \dot{y}}{\partial x} + q\frac{\partial \theta}{\partial x}\theta^{-1}.$$

Introducing in the above expression the Helmholtz free energy, ψ, which is defined per unit reference volume as

$$\psi = \varepsilon - \theta\eta,$$

one finally obtains that

$$\dot{\psi} \leq s\frac{\partial \dot{y}}{\partial x} + q\frac{\partial \theta}{\partial x}\theta^{-1} - \eta\dot{\theta}. \qquad (2.12)$$

The Helmholtz free energy is related to the amount of useful work. In a thermomechanical analysis Eqs. (2.3) and (2.6) as well as Eq. (2.12) are used to determine the motion $y(x,t)$ and the temperature $\theta(x,t)$. Indeed, the number of equations and unknowns do not coincide. It is necessary to develop the so called *constitutive equations* for the constitutive quantities that are: the stress s, the heat flux q, the free energy ψ and the entropy η. Constitutive equations are expressed as functionals. The constitutive equations relate the constitutive quantities to motion and temperature as well as their derivatives with respect to time and space.

The first issue to be argued is indeed the way in which the constitutive equations are established. The chapter *Constitutive Theories: Basic Principles* has focused on general requirements or principles that constitutive equations have to obey. It is clear that the behavior of a point of the body can only rest in general on what has happened to the body before the present time. This is known as the principle of determinism. One may develop a theory establishing that the behavior at a point x_0 at a given instant of time t_0 is only affected by the behavior of the points close to that point x_0 at times close to t_0. Whence, a Taylor series expansion of the motion about x_0, t_0 to first order in the variables x, t yields

$$y(x,t) \approx y(x_0,t_0) + \left|\frac{y}{x}\right|_{x=x_0,t=t_0}(x-x_0) + |\dot{y}|_{x=x_0,t=t_0}(t-t_0).$$

A similar expansion follows for the temperature. Hence, for this local interaction is sufficient to consider that the various constitutive equations are given as constitutive functionals of the following quantities:

$$y,\lambda,\dot{y},\theta,\frac{\partial\theta}{\partial x},\dot{\theta}.$$

It is further required that the values of the constitutive functions in the constitutive functional are unaffected by superposing rigid motions. This is known as material *objectivity* and mathematically means the following. Consider two motions y and $\bar{y}$ which obey

$$\bar{y}(x,t) = y(x,t) + g(t)$$

where g is an arbitrary smooth function. The main idea is that the constitutive functions in the constitutive functional are equal for both motions (provided that the temperature fields θ and $\bar{\theta}$ are equal). This establishes that the constitutive equations cannot depend on y and $\dot{y}$ since both are not *objectives* quantities.

Let us use now the Second Law of Thermodynamics or Clausius-Duhem inequality. Consider that the constitutive equations depend on the objective quantities

$$\lambda, \theta, \frac{\partial \theta}{\partial x}, \dot{\theta}, \tag{2.13}$$

and let us manipulate the inequality Eq. (2.12) to write it as

$$\left(\frac{\partial \psi}{\partial \lambda} - s\right)\dot{\lambda} + \left(\frac{\partial \psi}{\partial \theta} + \eta\right)\dot{\theta} + \frac{\partial \psi}{\partial \dot{\theta}}\ddot{\theta} + \frac{\partial \psi}{\partial(\partial \theta / \partial x)}\frac{\partial \dot{\theta}}{\partial x} \leq q\frac{\partial \theta}{\partial x}\theta^{-1}.$$

Let us fix the variables in Eq. (2.13) at $x = x_0$ and $t = t_0$. The variables $\dot{\lambda}$, $\ddot{\theta}$ and $\partial \dot{\theta} / \partial x$ are independent. Now, considering

$$\psi = \psi(\lambda, \theta), \; s = \frac{\partial \psi}{\partial \lambda},$$

the last inequality yields

$$q\frac{\partial \theta}{\partial x}\theta^{-1} - \left(\frac{\partial \psi}{\partial \theta} + \eta\right)\dot{\theta} \geq 0.$$

Assuming further that η is independent of $\dot{\theta}$ and $\partial \theta / \partial x$ and that q is independent of $\dot{\theta}$, it follows that $\eta = -\frac{\partial \psi}{\partial \theta}$ and the last inequality can be written as

$$q\left(\lambda, \theta, \frac{\partial \theta}{\partial x}\right)\frac{\partial \theta}{\partial x} \geq 0.$$

This last form of the Clausius-Duhem inequality is possible in the first place due to the local theory assumption between the different constitutive quantities and the motion and temperature. In a more general setting it is not always clear how to exploit the second law of thermodynamics.

Consider just a pure mechanical theory (temperature does not play any role). The linear momentum equation is now given by

$$\rho_0 \ddot{y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial \lambda}\right) + \rho_0 f, \tag{2.14}$$

or in Eulerian description

$$\rho\frac{d\mathbf{v}}{dt} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial \lambda}\right) + \rho f. \tag{2.15}$$

In Eq. (2.15) it is understood that the functions are expressed in terms of y. Whence, the constitutive formulation is complete with the free-energy function ψ.

It must be clear that the equations under study are not the equations governing one-dimensional structures as strings or beams. These are the governing equations for one dimensional motion of a continuum material.

A solution of this problem must obey the equations we have obtained (in a thermomechanical analysis in the Lagrangian description Eqs. (2.3) and (2.6) as well as Eq. (2.12)) with prescribed boundary conditions and initial conditions for the variables involved (motion and temperature). Next section will deal with important issues related to the existence of solutions.

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Biographical Sketches

Jose Merodio gained his first degree in Mechanical Engineering at Universidad del País Vasco, Spain. He completed his PhD in Engineering Mechanics and a MS in Applied Mathematics at Michigan State University, USA. Currently, he is an Associate Professor in the Department of Continuum Mechanics and

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Giuseppe Saccomandi is full Professor of Mathematical Physics at the School of Engineering of the University of Perugia. He was educated at the University of Perugia and he has held positions at the University of Roma La Sapienza and the University of Lecce (now of Salento). He has held visiting appointments at University Paris VI, University of Virginia and Universitat Politècnica de Catalunya. His research interests are in rational mechanics (mechanics of continua and nonlinear elasticity), applied mathematics and biomechanics in which fields he has authored or co-authored four invited book chapters and more than 130 publications in refereed archival journals.

Saccomandi is a member of the Gruppo Nazionale di Fisica Matematica of Istituto Nazionale di Alta Matematica and the International Society for the Interaction of Mechanics and Mathematics (ISIMM). He also coordinated three courses at the International Centre for Mechanical Sciences (CISM) of Udine (Topics in Finite Elasticity, 2000; Mechanics and Thermomechanics of Rubber-like Solids, 2002; Nonlinear Waves in Prestressed Materials, 2006) and edited three Springer volumes containing the lectures notes of these courses. He has served as guest editor for three special issues of the International J. of Nonlinear Mechanics, two special issues of IMA J. of Applied Mathematics and one special issue of Biomechanics and Modeling in Mechanobiology . He is on the editorial board of Mathematics and Mechanics of Solids and of the International Journal of Engineering Science. He has organized several conferences and mini-workshops as for example three mini-Workshop held at Mathematisches Forschungsinstitut Oberwolfach in 2002, 2006 and 2008 and he has organized the Istituto Nazionale di Alta Matematica Conference: Mathematical Problems in Elastodynamics and Related Continuum Theory held in Cortona in 2003. He has coordinated several research project as the project of national interest (PRIN 2004) Modeli Matematici per la Dinamica del DNA.