BALANCE LAWS

Anurag Gupta
Department of Mechanical Engineering, Indian Institute of Technology, Kanpur, UP 208016, India

David J. Steigmann
Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA

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Contents

Summary

Central to any theory of continuum mechanics, are balance laws of mass, momentum, and energy. These provide us with universal relations, which should be satisfied during every process associated with the continuous body. However, to obtain general statements of these laws, several integral theorems are required. A major portion of this chapter therefore deals with divergence theorem, Stokes' theorem and transport theorems, which are then used to obtain the balance laws in the form of partial differential equations to be satisfied away from the singular surface and jump conditions at the singular surface.

In this chapter we use, unless specified otherwise, the notation introduced in the previous chapter on kinematics. In particular, $\mathcal{E}$ denotes a three dimensional Euclidean space, associated with which is its translation space $\mathcal{V}$, a three dimensional inner product vector space. We use in addition, the subscript κ or χ when referring to a reference or a spatial frame, respectively. Let $\kappa(\mathfrak{B}) \subset \mathcal{E}$ and $\chi(\mathfrak{B}) \subset \mathcal{E}$ denote respectively, the fixed reference configuration and the current configuration. Let (t_1, t_2) be a fixed time interval, where $\{t_1, t_2\} \in \mathbb{R}$.

Integral Theorems

In this subsection we state and prove the localization theorem, the divergence theorem, the Stokes' theorem, and the transport theorem for volume and surface integrals. We have employed only elementary concepts from differential geometry in proving these theorems.

1. Localization Theorem for Volume Integrals

Let ϕ be a continuous function defined on an open set $R \subset \mathcal{E}$. If for all closed sets $\pi \subset R$

$$\int_{\pi} \phi dV = 0, \tag{1}$$

then $\phi(\mathbf{u}) = 0$ for all $\mathbf{u} \in R$. To prove this, we start by defining

$$I_{\varepsilon} = \left| \phi(\mathbf{u}_0) - \frac{1}{V_{\varepsilon}} \int_{s_{\varepsilon}} \phi(\mathbf{u}) dV \right| = \left| \frac{1}{V_{\varepsilon}} \int_{s_{\varepsilon}} (\phi(\mathbf{u}_0) - \phi(\mathbf{u})) dV \right|, \tag{2}$$

where s_{ε} is a sphere of radius ε and volume V_{ε} centered at $\mathbf{u}_0 \in R$. A theorem in analysis (Rudin, W. *Principles of Mathematical Analysis*, 3rd Ed., McGraw-Hill (1976), page 317) yields,

$$\begin{aligned} I_{\varepsilon} &\leq \frac{1}{V_{\varepsilon}} \int_{s_{\varepsilon}} |\phi(\mathbf{u}_0) - \phi(\mathbf{u})| \, dV \\ &\leq \frac{1}{V_{\varepsilon}} \int_{s_{\varepsilon}} \mathbf{u} \in s_{\varepsilon} \sup |\phi(\mathbf{u}_0) - \phi(\mathbf{u})| \, dV \\ &= \max_{\mathbf{u} \in s_{\varepsilon}} |\phi(\mathbf{u}_0) - \phi(\mathbf{u})|, \end{aligned} \tag{3}$$

where in $(3)_2$, sup can be replaced by max due to continuity and compactness of s_{ε}. Since $\phi(\mathbf{u})$ is continuous, we get $I_{\varepsilon} \to 0$ as $\varepsilon \to 0$. It then follows from Eq. (2),

$$\phi(\mathbf{u}_0) = \varepsilon \to 0 \lim \frac{1}{V_{\varepsilon}} \int_{s_{\varepsilon}} \phi(\mathbf{u}) dV = 0, \tag{4}$$

where the last equality is a consequence of (1). The point $\mathbf{u}_0$ can be chosen arbitrarily, and thus we can conclude that $\phi(\mathbf{u}) = 0$ for all $\mathbf{u} \in R$.

Localization Theorem for Surface Integrals Let φ be a continuous function defined on a surface $\mathcal{F} \subset \mathcal{E}$. If for all surfaces $\varsigma \subset \mathcal{F}$

$$\int_{\varsigma} \phi dA = 0, \tag{5}$$

then $\varphi(\mathbf{u}) = 0$ for all $\mathbf{u} \in \mathcal{F}$. This can be proved using arguments similar to those used above.

Divergence Theorem for Smooth Fields Let f, $\mathbf{p}$ and $\mathbf{P}$ be respectively, scalar, vector and tensor fields defined on $\kappa(\mathfrak{B})\times(t_1,t_2)$. Assume these fields to be continuously differentiable over $\kappa(\mathfrak{B})$. Then for any part $\Omega\subset\kappa(\mathfrak{B})$ and at any time $t\in(t_1,t_2)$

$$\int_\Omega\left(\nabla f\right)dV=\oint_{\partial\Omega}f\,\mathbf{N}\,dA \tag{6}$$

$$\int_\Omega\left(\mathrm{Div}\,\mathbf{p}\right)dV=\oint_{\partial\Omega}\mathbf{p}.\mathbf{N}dA \tag{7}$$

$$\int_\Omega\left(\mathrm{Div}\,\mathbf{P}\right)dV=\oint_{\partial\Omega}\mathbf{P}\,\mathbf{N}dA \tag{8}$$

where $\mathbf{N}\in\mathcal{V}_\kappa$ is the outward unit normal to the boundary $\partial\Omega$ of Ω. We outline a brief proof for (7). Let $\{\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3\}\in\mathcal{V}_\kappa$ be an orthonormal basis for $\mathcal{V}_\kappa$. Therefore there exists $\{p_1,p_2,p_3,X_1,X_2,X_3\}\in\mathbb{R}$ such that $\mathbf{p}=p_i\mathbf{E}_i$ and $\mathbf{X}=X_i\mathbf{E}_i$, with $i\in\{1,2,3\}$. Consider a cuboid $\mathcal{R}=\{\mathbf{X}\in\mathcal{E}_\kappa : A<X_1<B, C<X_2<D, E<X_3<F\}$, where $\{A,B,C,D,E,F\}\in\mathbb{R}$ are constants. Then the surface integral in (7), when written for the two faces of the cuboid which are orthogonal to $\mathbf{E}_1$, is

$$\begin{aligned}&\int_E^F\int_C^D(p_1(B,Y,Z)-p_1(A,Y,Z))dX_2dX_3\\&=\int_E^F\int_C^D\int_A^B\frac{\partial p_1}{\partial X_1}dX_1dX_2dX_3,\end{aligned} \tag{9}$$

which is obtained using the fundamental theorem of calculus (Rudin, W. *ibid.*, page 134). We can write similar relations for the surfaces of the cuboid orthogonal to $\mathbf{E}_2$ and $\mathbf{E}_3$. We get

$$\oint_{\partial\mathcal{R}}\mathbf{p}\;.\mathbf{N}\,dA=\int_{\mathcal{R}}\left(\frac{\partial p_1}{\partial X_1}+\frac{\partial p_2}{\partial X_2}+\frac{\partial p_3}{\partial X_3}\right)dV=\int_{\mathcal{R}}\left(\mathrm{Div}\,\mathbf{p}\right)dV. \tag{10}$$

We have therefore proved the divergence theorem for a cuboidal region. Furthermore, we can show that it holds for regions which are obtained by smooth deformations of the cuboid and also for general regions which can be obtained by pasting together the deformed cuboids (This argument can be found in the elementary texts on calculus. For a more advanced treatment see Rudin, W. *ibid.*, page 288).

Equation (6) is obtained from (7) for a scalar $\mathbf{p}$. A proof for (8) also follows from (7). Indeed, for an arbitrary constant $\mathbf{a}\in\mathcal{V}_\kappa$,

$$\mathbf{a}.\oint_{\partial\Omega}\mathbf{P}\,\mathbf{N}\,dA=\oint_{\partial\Omega}\left(\mathbf{P}^{\mathrm{T}}\mathbf{a}\right).\mathbf{N}\,dA=\int_\Omega\left(\mathrm{Div}\,\mathbf{P}^{\mathrm{T}}\mathbf{a}\right)dV=\int_\Omega\left(\mathrm{Div}\,\mathbf{P}\right).\mathbf{a}\,dV, \tag{11}$$

where in the last equality, the definition of the Div operator has been used. Since $\mathbf{a}$ is arbitrary, we get the desired result.

Divergence Theorem for Piecewise Smooth Fields Assume $\mathbf{p}$ to be piecewise continuously differentiable over $\kappa(\mathfrak{B})$, being discontinuous across the singular surface S_t (with normal $\mathbf{N}_s$ and speed U) and smooth everywhere else. Then for a domain Ω such that $\mathcal{S}=\Omega\cap S_t\neq\varnothing$,

$$\oint_{\partial\Omega}\mathbf{p}.\mathbf{N}\,dA=\int_{\Omega}(\mathrm{Div}\,\mathbf{p})\,dV+\int_{\mathcal{S}}[\![\mathbf{p}]\!].\mathbf{N}_s dA. \qquad (12)$$

Similar statements hold for scalar and tensor fields. We now prove (12). Let $\Omega^{\pm}\subset\Omega$ be such that $\Omega^{+}\cup\Omega^{-}=\Omega$ and $\Omega^{+}\cap\Omega^{-}=S$. The normal to the surface S is oriented such that it points into Ω^{+}. Since $\mathbf{p}$ is smooth within Ω^{+} and Ω^{-}, we can use (7) to write

$$\int_{\Omega^{+}}(\mathrm{Div}\,\mathbf{p})dV=\int_{\partial\Omega^{+}\backslash\mathcal{S}}\mathbf{p}\cdot\mathbf{N}dA-\int_{\mathcal{S}}\mathbf{p}^{+}\cdot\mathbf{N}_s dA,$$

$$\int_{\Omega^{-}}(\mathrm{Div}\,\mathbf{p})dV=\int_{\partial\Omega^{-}\backslash\mathcal{S}}\mathbf{p}\cdot\mathbf{N}dA+\int_{\mathcal{S}}\mathbf{p}^{-}\cdot\mathbf{N}_s dA,$$

where $\mathbf{p}^{\pm}$ are the limiting values of $\mathbf{p}$ as it approaches $\mathcal{S}$ from the interior of $\Omega^{\pm}$. The relation (12) is obtained by adding these two equations.

If $\mathbf{q}$ is a vector field defined on $\chi(\mathfrak{B})\times(t_1,t_2)$ and piecewise continuously differentiable over $\chi(\mathfrak{B})$, being discontinuous across the singular surface s_t (with normal $\mathbf{n}_s$ and speed u). Then for $\omega\subset\chi(\mathfrak{B})$ such that $s=\omega\cap s_t\neq\varnothing$,

$$\oint_{\partial\omega}\mathbf{q}.\mathbf{n}\,da=\int_{\omega}(\mathrm{div}\,\mathbf{q})\,dv+\int_{\mathcal{S}}[\![\mathbf{q}]\!].\mathbf{n}_s da. \qquad (13)$$

The proof for (13) is similar to that of (12).

Stokes' Theorem for Smooth Fields Let $\mathbf{p}$ and $\mathbf{P}$ be respectively, vector and tensor fields defined on $\kappa(\mathfrak{B})\times(t_1,t_2)$. Assume these fields to be continuously differentiable over $\kappa(\mathfrak{B})$. Then for any surface $\mathcal{F}\subset\kappa(\mathfrak{B})$ with normal $\mathbf{N}$ and boundary $\partial\mathcal{F}$

$$\int_{\mathcal{F}}(\mathrm{Curl}\,\mathbf{p}).\mathbf{N}dA=\oint_{\partial\mathcal{F}}\mathbf{p}\,.\,d\mathbf{X}, \qquad (14)$$

$$\int_{\mathcal{F}}(\mathrm{Curl}\,\mathbf{P})^{\mathrm{T}}\,\mathbf{N}dA=\oint_{\partial\mathcal{F}}\mathbf{P}\,d\mathbf{X}. \qquad (15)$$

A proof for (14) can be obtained from (Rudin, W. *ibid.*, page 287). To verify (15), we use (14). Indeed, for an arbitrary constant vector $\mathbf{a}\in\mathcal{V}_{\kappa}$,

$$\mathbf{a}\,.\int_{\mathcal{F}}(\mathrm{Curl}\,\mathbf{P})^{\mathrm{T}}\,\mathbf{N}dA=\int_{\mathcal{F}}(\mathrm{Curl}\,\mathbf{P}^{\mathrm{T}}\mathbf{a}).\mathbf{N}dA=\mathbf{a}\,.\oint_{\partial\mathcal{F}}\mathbf{P}\,d\mathbf{X}, \qquad (16)$$

where in the first equality, the definition of the *Curl* of a tensor field is used. The desired result follows upon using the arbitrariness of $\mathbf{a}$.

Stokes' Theorem for Piecewise Smooth Fields Consider $\mathbf{p}$ to be piecewise continuously differentiable over $\kappa(\mathfrak{B})$. Assume $\mathbf{p}$ to be discontinuous across the singular surface S_t and smooth everywhere else. Let $\Gamma = \mathcal{F} \cap S_t$ be the curve of intersection. Then

$$\int_{\mathcal{F}} (\text{Curl}\,\mathbf{p}) \,.\, \mathbf{N}\, dA = \oint_{\partial\mathcal{F}} \mathbf{p} \,.\, d\mathbf{X} + \int_{\Gamma} [\![\mathbf{p}]\!] \,.\, d\mathbf{X}, \tag{17}$$

To verify this relation start by considering two subsurfaces $\mathcal{F}^{\pm} \subset \mathcal{F}$ such that $\mathcal{F}^{+} \cup \mathcal{F}^{-} = \mathcal{F}$ and $\mathcal{F}^{+} \cap \mathcal{F}^{-} = \Gamma$. Since $\mathbf{p}$ is smooth in regions $F^{\pm}$, we can write using (14)

$$\int_{\mathcal{F}^{+}} (\text{Curl}\,\mathbf{p}) \cdot \mathbf{N} dA = \int_{\partial\mathcal{F}^{+}\setminus\Gamma} \mathbf{p} \cdot d\mathbf{X} + \int_{\Gamma} \mathbf{p}^{+} \cdot d\mathbf{X},$$
$$\int_{\mathcal{F}^{-}} (\text{Curl}\,\mathbf{p}) \cdot \mathbf{N} dA = \int_{\partial\mathcal{F}^{-}\setminus\Gamma} \mathbf{p} \cdot d\mathbf{X} - \int_{\Gamma} \mathbf{p}^{-} \cdot d\mathbf{X}.$$

Adding these two relations we get (17). Similarly, we obtain for a piecewise continuously differentiable tensor field $\mathbf{P}$:

$$\int_{\mathcal{F}} (\text{Curl}\,\mathbf{P})^{T}\, \mathbf{N} dA = \oint_{\partial\mathcal{F}} \mathbf{P}\, d\mathbf{X} + \int_{\Gamma} [\![\mathbf{P}]\!] d\mathbf{X}, \tag{18}$$

If $\mathbf{q}$ is a piecewise continuously differentiable vector field defined on $\chi(\mathfrak{B}) \times (t_1, t_2)$, being discontinuous across the singular surface s_t. Consider a surface $F \subset \chi(\mathfrak{B})$ with normal $\mathbf{n}$ and let $\gamma = F \cap s_t$. Then

$$\int_{F} (\text{curl}\,\mathbf{q}) . \mathbf{n} da = \oint_{\partial F} \mathbf{q} \,.\, d\mathbf{X} + \int_{\gamma} [\![\mathbf{q}]\!] . d\mathbf{X}, \tag{19}$$

The proof for (19) is similar to that of (17).

Remark (Surface divergence theorem) Consider a vector field p continuously differentiable over the surface $S \subset \kappa(\mathfrak{B})$ (with unit normal $\mathbf{N}$ and mean curvature H) for a fixed time interval (t_1, t_2). Then

$$\oint_{\partial S} \mathrm{p} \,.\, \nu dL = \int_{S} \left(\text{Div}^{S}\, \mathrm{p} + 2H\, \mathrm{p} . \mathbf{N} \right) dA, \tag{20}$$

where ν is the outer unit normal to ∂S such that $(\mathbf{N}, \nu, \mathbf{t})$ form a positively-oriented orthogonal triad at ∂S with $\mathbf{t}$ being the tangent vector along ∂S. Moreover, if p is tangential, i.e. $\mathbb{P}\mathrm{p} = \mathrm{p}$, then $\mathrm{p} \cdot \mathbf{N} = 0$ and (20) reduces to

$$\oint_{\partial S} \mathrm{p} \,.\, \nu dL = \int_S \mathrm{Div}^S \, \mathrm{p} \, dA. \tag{21}$$

We now prove (20). By definition $\nu = \mathbf{t} \times \mathbf{N}$ and therefore we can use Stokes' theorem to rewrite the term on the left hand side of Eq. (20) as

$$\begin{aligned} \oint_{\partial S} \mathrm{p} \,.\, \nu dL &= \oint_{\partial S} \mathrm{p} \,.\, (\mathbf{t} \times \mathbf{N}) dL, \\ &= \oint_{\partial S} (\mathbf{N} \times \mathrm{p}) \,.\, \mathbf{t} \, dL \\ &= \int_S \mathrm{Curl}(\mathbf{N} \times \mathrm{p}) \,.\, \mathbf{N} dA \end{aligned} \tag{22}$$

Use the identity $\mathrm{Curl}(\mathbf{N} \times \mathrm{p}) = \mathrm{Div}(\mathbf{N} \otimes \mathrm{p} - \mathrm{p} \otimes \mathbf{N})$ to get

$$\mathrm{Curl}(\mathbf{N} \times \mathrm{p}) \cdot \mathbf{N} = (\nabla \mathbf{N})^T \mathbf{N} \cdot \mathrm{p} - (\mathrm{p} \cdot \mathbf{N}) \, \mathrm{Div} \, \mathbf{N} + \nabla \, \mathrm{p} \cdot \mathbb{P}. \tag{23}$$

But $(\nabla \mathbf{N})^T \mathbf{N} = 0$ (follows from $\mathbf{N} \cdot \mathbf{N} = 1$) and $\nabla \, \mathrm{p} \cdot \mathbb{P} = \mathrm{tr}(\nabla \, \mathrm{p} \, \mathbb{P}) = \mathrm{Div}^S \, \mathrm{p}$. Furthermore, $\mathrm{Div} \, \mathbf{N} = -2H$ (using a result from the chapter on kinematics). Therefore we can rewrite (23) to get

$$\mathrm{Curl}(\mathbf{N} \times \mathrm{p}) \cdot \mathbf{N} = 2H(\mathrm{p} \cdot \mathbf{N}) + \mathrm{Div}^S \, \mathrm{p}. \tag{24}$$

Substituting this into (22) yields (20).

Transport theorem for volume integrals with smooth fields Let P and Q denote a scalar, vector or tensor field continuously differentiable on $\kappa(\mathfrak{B}) \times (t_1, t_2)$ and $\chi(\mathfrak{B}) \times (t_1, t_2)$, respectively. Then for arbitrary parts $\Omega \subset \kappa(\mathfrak{B})$, $\omega \subset \chi(\mathfrak{B})$ and at any time $t \in (t_1, t_2)$

$$\frac{d}{dt} \int_\Omega P dV = \int_\Omega \dot{P} dV, \tag{25}$$

$$\frac{d}{dt} \int_\omega Q dv = \int_\omega \frac{\partial Q}{\partial t} dv + \int_{\partial \omega} Q(\mathbf{v} \cdot \mathbf{n}) da. \tag{26}$$

Since Ω is fixed with respect to time and P is smooth over Ω, the time derivative and the volume integral in the left hand side of (25) can commute to give the right hand side of the equation. Equation (26) can be proved by first transforming the volume ω to a fixed reference volume, say Ω. We get

$$\begin{aligned} \frac{d}{dt} \int_\omega Q \, dv &= \frac{d}{dt} \int_\Omega Q \, J \, dV \\ &= \int_\omega \dot{Q} \, dv + \int_\omega Q(\mathrm{div} \, \mathbf{v}) dv, \end{aligned} \tag{27}$$

where J is the Jacobian associated with the mapping which transforms Ω to ω and $\dot{J} = J(\operatorname{div}\mathbf{v})$. Equation (26) follows from (27) upon recalling the definition of the material time derivative and using the divergence theorem.

Transport theorem for volume integrals with piecewise smooth fields Let Ω be such that $\mathcal{S} = \Omega \cap S_t \neq \varnothing$. Then for a P which is discontinuous across S_t but smooth everywhere else,

$$\frac{d}{dt}\int_\Omega P dV = \int_\Omega \dot{P} dV - \int_{\mathcal{S}} U [\![P]\!] dA. \tag{28}$$

We now prove this relation. Recall surface parameterization introduced at the end of the chapter on kinematics. In a small neighborhood, say $\Omega_{\mathcal{S}}$, of the singular surface $\mathcal{S}$ we parameterize the domain by coordinates $\{\xi_1, \xi_2, \zeta\}$ such that for $\mathbf{X} \in \Omega_S$ we can write $\mathbf{X} = \hat{\mathbf{X}}(\xi_1, \xi_2, t) + \zeta(t)\mathbf{N}(\xi_1, \xi_2, t)$, where $\hat{\mathbf{X}} \in \mathcal{S}$ and $\{\xi_1, \xi_2\}$ are convected. Let $-\varsigma < \zeta(t) < \varsigma$, where $\varsigma \in \mathbb{R}^+$ is constant. The position of the singular surface is indicated by $\zeta = 0$ and it is assumed that the surface $\mathcal{S}$ remains inside $\Omega_{\mathcal{S}}$ during the instantaneous motion. Obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega P dV &= \frac{d}{dt}\int_{\Omega\setminus\Omega_{\mathcal{S}}} P dV + \frac{d}{dt}\int_{\Omega_{\mathcal{S}}} P dV \\
&= \int_{\Omega\setminus\Omega_{\mathcal{S}}} \dot{P} dV + \int_{(\xi_1,\xi_2)} \frac{d}{dt}\left(\int_{-\varsigma}^{\varsigma} P j_A d\zeta\right) dA_\xi, \\
&= \int_{\Omega\setminus\Omega_{\mathcal{S}}} \dot{P} dV \\
&\quad + \int_{(\xi_1,\xi_2)} \left\{ \frac{d}{dt}\left(\int_{-\varsigma}^{\zeta_1(t)} P j_A d\zeta + \int_{\zeta_2(t)}^{\varsigma} P j_A d\zeta\right)\right\}_{\zeta_1,\zeta_2=0} dA_\xi,
\end{aligned}$$

where j_A is the Jacobian related to the change of coordinates. On the singular surface, $\zeta_1 = \zeta_2 = 0$, $\dot{\zeta}_1 = \dot{\zeta}_2 = U$, $j_A = \xi$ and $dA = \xi dA_\xi$, where ξ is the surface Jacobian. Taking the limit $|\varsigma| \to 0$ we obtain the desired result. The infinitesimal area of the surface in terms of the new coordinates can be obtained by using Nanson's formula, $\mathbf{N}dA = j_A \mathbf{A}^{-\mathrm{T}}\hat{\mathbf{N}} dA_\xi$, where $\hat{\mathbf{N}} = \mathbf{N}$ and $\mathbf{A}$ is the gradient of the map from the new coordinates to $\mathbf{X}$. For the considered transformation this formula reduces to $dA = j_A dA_\xi$.

Let ω be such that $s = \omega \cap s_t \neq \varnothing$. Then for a Q which is discontinuous across s_t but smooth everywhere else,

$$\frac{d}{dt}\int_\omega Q dv = \int_\omega \left(\frac{\partial Q}{\partial t} + \operatorname{div}(Q\mathbf{v})\right) dv - \int_s \left(u[\![Q]\!] - [\![Q\mathbf{v}]\!] \cdot \mathbf{n}\right) da, \tag{29}$$

where $u = U\,|(\mathbf{F}^{\pm})^{\mathrm{T}}\mathbf{n}| + \mathbf{n}\cdot\mathbf{v}^{\pm}$ is the spatial speed of the singular surface s_t. This relation can be proved by first transforming ω to Ω and then using (28). We get

$$\frac{d}{dt}\int_{\omega} Q dv = \int_{\Omega} (JQ)^{\cdot} dV - \int_{S} U [\![JQ]\!] dA. \tag{30}$$

The term $U[\![JQ]\!]$ can be expanded as

$$\begin{aligned} U[\![JQ]\!] &= \left(Q^{+}u^{+}J^{+}\,|(\mathbf{F}^{+})^{-\mathrm{T}}\mathbf{N}|\right) - \left(Q^{-}u^{-}J^{-}\,|(\mathbf{F}^{-})^{-\mathrm{T}}\mathbf{N}|\right) \\ &= \left(u[\![Q]\!] - [\![Q\mathbf{v}]\!]\cdot\mathbf{n}\right)|(\mathbf{F}^{-})^{*}\mathbf{N}|, \end{aligned} \tag{31}$$

where $u^{\pm} = u - \mathbf{n}\cdot\mathbf{v}^{\pm}$. Relations $U = u^{\pm}\,|(\mathbf{F}^{\pm})^{-\mathrm{T}}\mathbf{N}|$ and $|(\mathbf{F}^{+})^{*}\mathbf{N}| = |(\mathbf{F}^{-})^{*}\mathbf{N}|$ have also been used. Equation (29) follows immediately after substituting (31) into (30).

Transport theorem for surface integrals with smooth fields Let p be a scalar, vector or tensor field continuously differentiable on $S_t \times (t_1, t_2)$. Then, for an arbitrary surface $\mathcal{S} \subset S_t$

$$\frac{d}{dt}\int_{\mathcal{S}} \mathrm{p}\, dA = \int_{\mathcal{S}} (\dot{\mathrm{p}} - 2\,\mathrm{p}UH) dA, \tag{32}$$

where $\mathbf{N}$, U, and H are the unit normal, normal velocity, and the mean curvature associated with S_t, respectively. We prove this relation using the surface parameterization outlined in the chapter on kinematics. We assume that p can be extended to the small neighborhood $\Omega_{\mathcal{S}}$, and use the same symbol to denote its extension. Obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{S}} \mathrm{p}\, dA &= \left\{\frac{d}{dt}\int_{(\xi_1,\xi_2)} \mathrm{p}\left(\mathbf{X}(\xi_\alpha, \zeta(t)), t\right) j_A dA_\xi\right\}_{\zeta=0} \\ &= \int_{(\xi_1,\xi_2)} \left\{(\dot{\mathrm{p}}\, j_A + j_A\dot{\zeta}(\nabla\mathrm{p}\cdot\mathbf{N}) + \mathrm{p}\,\dot{j}_A) dA_\xi\right\}_{\zeta=0} \\ &= \int_{\mathcal{S}} \left\{\dot{\mathrm{p}} + \dot{\zeta}\,\nabla\mathrm{p}\cdot\mathbf{N} + \mathrm{p}\,\dot{j}_A j_A^{-1}\right\}_{\zeta=0} dA. \end{aligned} \tag{33}$$

At the surface, $\zeta = 0$, we have $\dot{\zeta} = U$, $j_A = \xi$ and $\dot{j}_A = -2UH\xi$. Substituting these into (33) and recalling the definition of the normal time derivative, we obtain (32).

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Biographical Sketches

Anurag Gupta He received B.Tech. in Civil Engineering from Indian Institute of Technology at Roorkee in 2002, M.S. in Civil and Environmental Engineering from University of California at Berkeley in 2003, and Ph.D. in Mechanical Engineering from University of California at Berkeley in 2008. His thesis dealt with plastic deformation in solids with interfaces. Currently, he is an Assistant Professor in the department of Mechanical Engineering at Indian Institute of Technology, Kanpur, India. His research interests include plasticity theory, dynamics of defects in solids, thermodynamics of irreversible processes, and thin films.

David J Steigmann He received B.S. in Aeronautics and Astronautics from University of Michigan at Ann Arbor in 1979, M.S. in Aeronautics and Astronautics from M.I.T. (Cambridge) in 1982, and Ph.D. in Applied Mathematics from Brown University, Providence in 1988. Currently he is a Professor in the department of Mechanical Engineering at University of California, Berkeley, USA. Before joining Berkeley, he was a Professor in the department of Mechanical Engineering at University of Alberta, Canada (till 1997). His research interests are in the following areas. Mechanics of thin films and thin-film/substrate systems: near-surface wave propagation and energy flux; Electromagnetic phenomena in solid mechanics: applications to thin-film/substrate problems; Surface stress, capillary phenomena, biological cell membranes, surfactant films in multi-phase fluid emulsions; Finite elasticity; Variational methods and elastic stability; Tensile (membrane) structures; Continuum mechanics; Nonlinear three-dimensional mechanics of fabrics; Numerical analysis of ill-conditioned structural problems; Thin shells.