SIMPLE CONSTITUTIVE MODELS FOR SOLIDS AND FLUIDS

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Contents

Summary

In this chapter we review the fundamentals and the formulation of some simple material constitutive models both for fluids and solids. First, a short review of the main concepts and definitions of Continuum Mechanics is presented, including kinematics and balance principles, in order to make this chapter auto-comprehensive and to fix the terminology and notation. Next, we include the fundamental principles of constitutive models, from a general perspective and without entering in details that have been presented elsewhere in this book, but sufficient to recall the concepts needed in the following sections. Then, we enter in the core of the chapter, discussing the fundamentals of elastic and hyperelastic materials, with especial emphasis in those with fibered microstructure, as well as plastic and viscoplastic solids, viscoelastic liquids and solids and, finally, of

damage-based constitutive models. For a better understanding, in some cases, we have added some illustrating examples and the computational aspects related.

As it is easy to understand, this cannot be a book on rheology, that would need a much longer space, but only an introduction to the main families of constitutive models for engineering materials. In any case, we have tried to be not too restrictive, so fully non-linear kinematics has been assumed in all cases. A much more detailed presentation may be found in the many references included in the chapter.

1. Basic Results in Continuum Mechanics

Below we summarize some basic results of nonlinear continuum mechanics relevant to our subsequent developments. For further details we refer to [11] [31] [32] or [25].

1.1. Kinematics

Let $\mathcal{B} \subset \mathbb{R}^3$ denote the *reference configuration* of a continuum body defined as a set of points located by their respective co-ordinates with respect to a fixed but otherwise arbitrary reference frame $\{\mathbf{X}\}$ at time $t = 0$, and with its particles labeled as $\mathbf{X} \in \mathcal{B}$. For our purposes, it suffices to regard $\mathcal{B}$ as an open bounded set in $\mathbb{R}^3$. A *smooth deformation* is a continuously differentiable one to one mapping (as well as its inverse):

$$\boldsymbol{\chi} : \mathcal{B} \to \mathcal{S} \subset \mathbb{R}^3 \tag{1}$$

which puts into correspondence $\mathcal{B} \subset \mathbb{R}^3$ with some region $\mathcal{S} \subset \mathbb{R}^3$, the *deformed configuration*, in the Euclidean space (see Figure 1). A *motion* is a one-parameter family of deformations, $\boldsymbol{\chi}_t$, parameterized by time, such as, for a fixed time t, (1) represents a deformation mapping between the undeformed and deformed bodies. On the contrary, for a fixed particle $\mathbf{X}$, (1) gives the trajectory of this particle as a function of time.
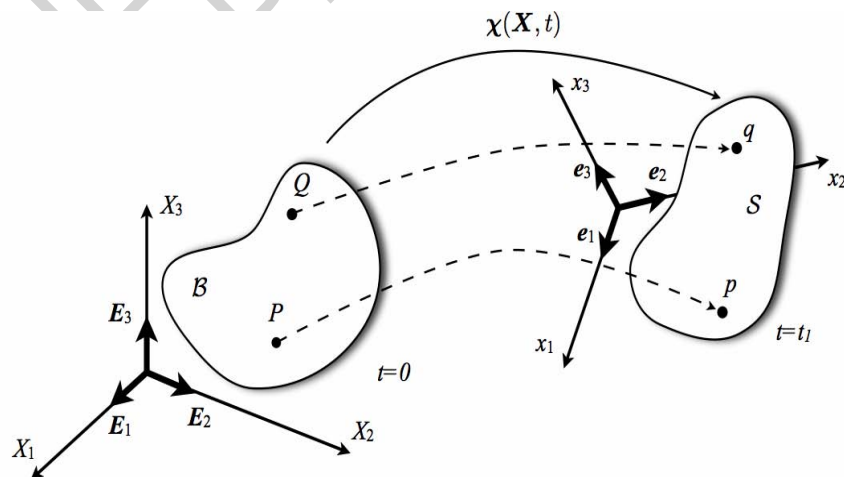

Figure 1. Motion of a deformable body.

In a deformable body, those properties which change along with the deformation of the body might be described either by the evolution of its value along the trajectory of a given material point, *material description* (also known as *Lagrangian description*), or by the change of its value at a fixed location in space occupied by (different for each time instant) particles of the body, *spatial* (*Eulerian*) *description*.

The *deformation gradient* is the derivative of the deformation mapping χ. We use the notation (From now on, all the newly defined functions that depend on the motion also depend on time. However, the reference to *t* will be systematically dropped, except when needed, making such dependence implicit for not complicating the notation.)

$$F\left(\mathbf{X}\right) = \nabla_{\mathbf{X}}\chi = \frac{\partial \chi\left(\mathbf{X}\right)}{\partial \mathbf{X}} = \sum_{i,I=1}^{3} F_{iI}\mathbf{e}_{i} \otimes \mathbf{E}_{I}. \tag{2}$$

The deformation gradient transforms vectors in the reference configuration to vectors in the current configuration, thus describing the change in the relative position of two material particles (or the vector that joins them), and is therefore a two-point tensor,

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}. \tag{3}$$

Similarly, by using the inverse of the deformation gradient, $\mathbf{F}^{-1}$, material vectors can be written in terms of the corresponding spatial vector as

$$d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}. \tag{4}$$

Expressions (3) and (4) are usually referred as *push-forward* and *pull-back* operations of the corresponding vectors, and are expressed, respectively, as $d\mathbf{x} = \chi \circ d\mathbf{X}$, and $d\mathbf{X} = \chi^{-1} \circ d\mathbf{x}$.

By using (3), the infinitesimal volume in the current configuration is given as

$$dv = d\mathbf{x}_{1} \cdot \left(d\mathbf{x}_{2} \times d\mathbf{x}_{3}\right) = JdV = Jd\mathbf{X}_{1} \cdot \left(d\mathbf{X}_{2} \times d\mathbf{X}_{3}\right), \tag{5}$$

with $J = \det \mathbf{F}$, the *Jacobian of the deformation*, such that the local condition of impenetrability of matter requires that the local volume ratio

$$J\left(\mathbf{X}\right) := \det\left[\mathbf{F}\left(\mathbf{X}\right)\right] > 0. \tag{6}$$

This result allows expressing the conservation of mass as follows

$$dm = \rho dv = \rho JdV = \rho_{0}dV \Rightarrow \rho_{0}\left(\mathbf{X}\right) = \rho\left(\mathbf{X}\right)J\left(\mathbf{X}\right). \tag{7}$$

with ρ the current density and ρ_{0} the initial density.

Let us consider now an area element in the material configuration, $d\mathbf{S} = dS\mathbf{N}$, and an arbitrary material vector $\mathbf{L}$ not orthogonal to $\mathbf{N}$. After deformation, these elements become $d\mathbf{s} = ds\mathbf{n}$, and $\mathbf{l} = \mathbf{FL}$. Therefore we can write

$$dv = d\mathbf{s} \cdot \mathbf{l} = d\mathbf{s} \cdot \mathbf{FL} = \mathbf{F}^{\mathbf{T}} d\mathbf{s} \cdot \mathbf{L}, \tag{8}$$

$$dV = d\mathbf{S} \cdot \mathbf{L} . \tag{9}$$

Using (5) we finally get

$$d\mathbf{s} = J\mathbf{F}^{-\mathbf{T}} d\mathbf{S} , \tag{10}$$

known as Nanson's formula.

Let us now consider the change in the inner product of two material vectors $d\mathbf{X}_1$ and $d\mathbf{X}_2$ as they deform through the motion. Using (3), we have

$$d\mathbf{x}_1 \cdot d\mathbf{x}_2 = d\mathbf{X}_1 \cdot \mathbf{F}^{\mathbf{T}}\mathbf{F} \cdot d\mathbf{X}_2 = d\mathbf{X}_1 \cdot \mathbf{C} \cdot d\mathbf{X}_2, \tag{11}$$

where $\mathbf{C} = \mathbf{F}^{\mathbf{T}}\mathbf{F}$ is the *right Cauchy-Green deformation tensor* (Note that $\mathbf{C}$ only operates on material quantities, and is therefore a material tensor.). Similarly, the inner product of two material vectors can be expressed as

$$d\mathbf{X}_1 \cdot d\mathbf{X}_2 = d\mathbf{x}_1 \cdot \mathbf{F}^{-\mathbf{T}}\mathbf{F}^{-1} \cdot d\mathbf{x}_2 = d\mathbf{x}_1 \cdot \mathbf{b}^{-1} \cdot d\mathbf{x}_2, \tag{12}$$

where $\mathbf{b} = \mathbf{FF}^{\mathbf{T}}$ is the left *Cauchy-Green* or *Finger tensor* (In this case, $\mathbf{b}$ only operates on spatial quantities and consequently is a spatial tensor.).

Alternative definitions of strain can be found in terms of the difference in the scalar product of vectors $d\mathbf{X}_1$ and $d\mathbf{X}_2$ in the spatial and material configurations,

$$\frac{1}{2}\left(d\mathbf{x}_1 \cdot d\mathbf{x}_2 - d\mathbf{X}_1 \cdot d\mathbf{X}_2\right) = d\mathbf{X}_1 \cdot \mathbf{E} \cdot d\mathbf{X}_2, \tag{13}$$

where,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right) \tag{14}$$

is the *Lagrange strain* tensor and $\mathbf{I}$ is the second order unit tensor in the initial configuration such as $\mathbf{I}d\mathbf{X} = d\mathbf{X}$. Alternatively, the *Almansi strain* tensor is obtained as

$$\frac{1}{2}\left(d\mathbf{x}_1 \cdot d\mathbf{x}_2 - d\mathbf{X}_1 \cdot d\mathbf{X}_2\right) = d\mathbf{x}_1 \cdot \mathbf{e} \cdot d\mathbf{x}_2, \tag{15}$$

with

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{1} - \mathbf{b}^{-1}\right). \tag{16}$$

being **1** the second order unit tensor in the current configuration.

From (13) and (15) we get

$$d\mathbf{x}_1 \cdot \mathbf{e} d\mathbf{x}_2 = d\mathbf{X}_1 \cdot \mathbf{E} d\mathbf{X}_2, \tag{17}$$

so that the *push-forward* and *pull-back* operations of these strain measures become:

$$\mathbf{e} = \mathbf{F}^{-\mathbf{T}}\mathbf{E}\mathbf{F}^{-1} = \chi \circ \mathbf{E}, \tag{18}$$

$$\mathbf{E} = \mathbf{F}^{\mathbf{T}}\mathbf{e}\mathbf{F} = \chi^{-1} \circ \mathbf{e}. \tag{19}$$

When working with incompressible or nearly incompressible materials, it is sometimes useful to consider the multiplicative split of **F** into dilatational and distortional (isochoric) parts. This decomposition reads

$$\mathbf{F} = J^{1/3}\overline{\mathbf{F}}. \tag{20}$$

From this, it is now possible to define the isochoric counterparts $\overline{\mathbf{C}}$ and $\overline{\mathbf{b}}$ of the right and left Cauchy-Green deformation tensors (Note that $\det\left(\overline{\mathbf{F}}\right) = 1$.)

$$\begin{aligned} \mathbf{C} &= \mathbf{F}^{\mathbf{T}}\mathbf{F} = J^{2/3}\overline{\mathbf{C}}, \quad \overline{\mathbf{C}} = \overline{\mathbf{F}}^{T}\overline{\mathbf{F}}, \\ \mathbf{b} &= \mathbf{F}\mathbf{F}^{\mathbf{T}} = J^{2/3}\overline{\mathbf{b}}, \quad \overline{\mathbf{b}} = \overline{\mathbf{F}}\,\overline{\mathbf{F}}^{\mathbf{T}}. \end{aligned} \tag{21}$$

According to the polar decomposition theorem [31][32], we recall that the deformation gradient at any point $\mathbf{X} \in \mathcal{B}$ can be decomposed as

$$\mathbf{F}(\mathbf{X}) = \mathbf{R}(\mathbf{X})\mathbf{U}(\mathbf{X}) = \mathbf{V}\left[\chi(\mathbf{X})\right]\mathbf{R}(\mathbf{X}), \tag{22}$$

where $\mathbf{R}(\mathbf{X})$ is a proper orthogonal tensor $\left(\mathbf{R}^{\mathbf{T}}\mathbf{R} = \mathbf{I}\right)$, called the *rotation tensor*, and $\mathbf{U}(\mathbf{X}), \mathbf{V}\left[\chi(\mathbf{X})\right]$ are symmetric positive-definite tensors called the *right and left stretch tensors*, respectively, and defined as:

$$\mathbf{C} = \mathbf{U}^{\mathbf{T}}\mathbf{U}, \quad \mathbf{b} = \mathbf{V}\mathbf{V}^{\mathbf{T}}, \tag{23}$$

It is immediate to demonstrate that with the definitions (22)(23), **R** is an orthogonal tensor.

$$\mathbf{R}^\mathbf{T}\mathbf{R} = \mathbf{U}^{-\mathbf{T}}\mathbf{F}^\mathbf{T}\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-\mathbf{T}}\mathbf{C}\mathbf{U}^{-1} = \mathbf{U}^{-\mathbf{T}}\left(\mathbf{U}^\mathbf{T}\mathbf{U}\right)\mathbf{U}^{-1} = \mathbf{I}, \tag{24}$$

and similarly for the left version of the stretch tensor.

Let now $I_A, (A = 1, 2, 3)$ be the principal invariants of $\mathbf{C}$ (or of $\mathbf{b}$), defined as

$$
\begin{aligned}
I_1(\mathbf{C}) &:= \mathbf{tr}[\mathbf{C}] = \mathbf{C} : \mathbf{1}, \\
I_2(\mathbf{C}) &:= \mathbf{tr}\,\mathbf{C}^{-1}\det\mathbf{C} = \frac{1}{2}\left(\mathbf{tr}[\mathbf{C}]^2 - \mathbf{tr}\left[\mathbf{C}^2\right]\right), \\
I_3(\mathbf{C}) &:= \det\mathbf{C} = \frac{1}{6}\left(\mathbf{tr}[\mathbf{C}]^3 - 3\mathbf{tr}[\mathbf{C}]\mathbf{tr}\left[\mathbf{C}^2\right] + 2\mathbf{tr}\left[\mathbf{C}^3\right]\right).
\end{aligned}
\tag{25}
$$

Since $\mathbf{C}$ is symmetric and positive-definite, by the spectral theorem [31], we can write

$$\mathbf{C} = \sum_{A=1}^{3} \lambda_A^2 \mathbf{N}^{(A)} \otimes \mathbf{N}^{(A)}, \left\|\mathbf{N}^{(A)}\right\| = 1, \tag{26}$$

where $\lambda_A^2 > 0$ are the eigenvalues of $\mathbf{C}$, solutions of the characteristic polynomial equation:

$$p\left(\lambda^2\right) = \lambda^6 - I_1\lambda^4 + I_2\lambda^2 - I_3 = 0, \tag{27}$$

and $\mathbf{N}^{(A)}$ are the associated principal directions fulfilling:

$$\mathbf{C}\mathbf{N}^{(A)} = \lambda_A^2 \mathbf{N}^{(A)}, (A = 1, 2, 3). \tag{28}$$

In (28), λ_A, $(A = 1, 2, 3)$, are the *principal stretches* along the *principal directions* $\mathbf{N}^{(A)}$. The push-forward of the principal directions, $\mathbf{N}^{(A)}$, is written as

$$\boldsymbol{\chi} \circ \mathbf{N}^{(A)} = \mathbf{F}\mathbf{N}^{(A)} = \lambda_A \mathbf{n}^{(A)}, \left\|\mathbf{n}^{(A)}\right\| = 1. \tag{29}$$

The vectors in the triad $\left\{\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}\right\}$ are called the *Eulerian principal directions* at $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) \in \mathcal{S}$, so that the spectral decomposition of $\mathbf{F}$ takes the form

$$\mathbf{F} = \sum_{A=1}^{3} \lambda_A \mathbf{n}^{(A)} \otimes \mathbf{N}^{(A)}. \tag{30}$$

From (23) and (28), the spectral decompositions of the right and left stretch tensors are given by

$$\mathbf{U} = \sum_{A=1}^{3} \lambda_A \mathbf{N}^{(A)} \otimes \mathbf{N}^{(A)}, \tag{31}$$

$$\mathbf{V} = \sum_{A=1}^{3} \lambda_A \mathbf{n}^{(A)} \otimes \mathbf{n}^{(A)}, \tag{32}$$

respectively, while the spectral decomposition of the rotation tensor takes the form

$$\mathbf{R} = \sum_{A=1}^{3} \mathbf{n}^{(A)} \otimes \mathbf{N}^{(A)}. \tag{33}$$

The *material velocity*, denoted by $\mathbf{V}(\mathbf{X},t)$, is the time derivative of the motion (Do not confuse the material velocity with the left stretch tensor; the context will make clear the particular object considered) :

$$\mathbf{V}(\mathbf{X},t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial t}. \tag{34}$$

Similarly, the *material acceleration* is defined as the time derivative of the material velocity:

$$\mathbf{A}(\mathbf{X},t) = \frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t} = \frac{\partial^2 \boldsymbol{\chi}(\mathbf{X},t)}{\partial t^2}. \tag{35}$$

The spatial or Eulerian description can be obtained from the material description by changing the independent variables from material to spatial coordinates of a particle. Accordingly, at any time $t \in [0,T]$, one defines the *spatial velocity* and *acceleration fields*, denoted by $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{a}(\mathbf{x},t)$, respectively, by the change of variable formula $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$:

$$\mathbf{v}(\boldsymbol{\chi}(\mathbf{X},t),t)) = \mathbf{V}(\mathbf{X},t), \mathbf{a}(\boldsymbol{\chi}(\mathbf{X},t),t)) = \mathbf{A}(\mathbf{X},t), \tag{36}$$

or, in compact form,

$$\mathbf{v} = \mathbf{V} \circ \boldsymbol{\chi}^{-1}, \text{ and } \mathbf{a} = \mathbf{A} \circ \boldsymbol{\chi}^{-1}, \tag{37}$$

$$\mathbf{V} = \mathbf{v} \circ \boldsymbol{\chi} \text{ and } \mathbf{A} = \mathbf{a} \circ \boldsymbol{\chi}, \tag{38}$$

where "o" denotes composition (Observe that (38)/(37) are not the push-forward/pull-back operations on the velocity and acceleration, but only the change of variable of those functions associated to the motion)

The material time derivative of a spatial object, such as the spatial velocity, function of the variables $(\mathbf{x},t) \in \mathcal{S} \subset \mathbb{R}^3 \times [0,T]$, is the time derivative holding the particle (not its

current position) *fixed*. For example, for the spatial velocity, we denote its material time derivative by $\dot{\mathbf{v}}(\mathbf{x},t)$. Then, by definition,

$$\dot{\mathbf{v}}(\mathbf{x},t)\big|_{\mathbf{x}=\chi(\mathbf{X},t)} := \frac{\partial}{\partial t}\left\{\mathbf{v}\left[\chi(\mathbf{X},t),t\right]\big|_{\mathbf{X}=\text{Fixed}}\right\} = \frac{\partial}{\partial t}\mathbf{V}(\mathbf{X},t)\big|_{\mathbf{X}=\text{Fixed}} = \mathbf{A}(\mathbf{X},t). \qquad (39)$$

Therefore, by definition of spatial acceleration,

$$\dot{\mathbf{v}}(\mathbf{x},t) = \mathbf{A}(\mathbf{X},t)\big|_{\mathbf{X}=\chi^{-1}(\mathbf{x})} = \mathbf{a}(\mathbf{x},t), \qquad (40)$$

i.e., the material time derivative of the spatial velocity field is the spatial acceleration. In general, if $\boldsymbol{\sigma}(\mathbf{x},t)$ is a spatial tensor field, by definition, its material time derivative, denoted by $\dot{\boldsymbol{\sigma}}(\mathbf{x},t)$, is obtained by the formula

$$\dot{\boldsymbol{\sigma}} := \left[\frac{\partial}{\partial t}(\boldsymbol{\sigma} \circ \chi)\right] \circ \chi^{-1} = \frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}}\boldsymbol{\sigma} \qquad (41)$$

while $\Sigma(\mathbf{X},t) = \frac{\partial \Sigma}{\partial t}$.

Some additional important material derivatives are the following:

$$\dot{\mathbf{F}} = \frac{\partial \mathbf{F}}{\partial t} = \frac{\partial}{\partial t}[\nabla_{\mathbf{X}}\chi] = \nabla_{\mathbf{X}}\left(\frac{\partial \chi}{\partial t}\right) = \nabla_{\mathbf{X}}\mathbf{V} = \mathbf{L}, \qquad (42)$$

where $\mathbf{V}$ is the material velocity field given by (34), and $\nabla_{\mathbf{X}}\mathbf{V}$ is called the *material velocity gradient*. By the chain rule,

$$\nabla_{\mathbf{X}}\mathbf{V} = \nabla_{\mathbf{X}}[\mathbf{v} \circ \chi] = \nabla_{\mathbf{x}}\mathbf{v}\nabla_{\mathbf{X}}\chi = \nabla_{\mathbf{x}}\mathbf{v}\mathbf{F} = \mathbf{l}\mathbf{F}. \qquad (43)$$

Combining (42) and (43), we arrive at the following expression for the *spatial velocity gradient* $\mathbf{l} = \nabla_{\mathbf{x}}\mathbf{v}$:

$$\mathbf{l} = \nabla_{\mathbf{x}}\mathbf{v} = \frac{\partial \mathbf{F}}{\partial t}\mathbf{F}^{-1} = \mathbf{L}\mathbf{F}^{-1} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \qquad (44)$$

The symmetric part of $\nabla_{\mathbf{x}}\mathbf{v}$, denoted by $\mathbf{d}$, is called the *spatial rate of deformation tensor*, and its skew-symmetric part is called the *spin* or *vorticity tensor*, denoted by $\mathbf{w}$. Thus

$$\mathbf{d} = \frac{1}{2}[\nabla_{\mathbf{x}}\mathbf{v} + \nabla_{\mathbf{x}}\mathbf{v}^{\mathbf{T}}] = \frac{1}{2}(\mathbf{l}+\mathbf{l}^{\mathbf{T}}), \qquad (45)$$

$$\mathbf{w} = \frac{1}{2}\left[\nabla_{\mathbf{x}}\mathbf{v} - \nabla_{\mathbf{x}}\mathbf{v}^{\mathbf{T}}\right] = \frac{1}{2}(\mathbf{l}-\mathbf{l}^{\mathbf{T}}), \qquad (46)$$

with

$$\mathbf{l} = \nabla_{\mathbf{x}} \mathbf{v} = \mathbf{d} + \mathbf{w}. \tag{47}$$

The following relationship is also useful:

$$\mathbf{D} = \dot{\mathbf{E}} = \frac{1}{2}\dot{\mathbf{C}} = \frac{1}{2}\left[\dot{\mathbf{F}}^{\mathbf{T}}\mathbf{F} + \mathbf{F}^{\mathbf{T}}\dot{\mathbf{F}}\right] = \frac{1}{2}\mathbf{F}^{\mathbf{T}}[(\nabla_{\mathbf{x}}\mathbf{v})^{\mathbf{T}} + \nabla_{\mathbf{x}}\mathbf{v}]\mathbf{F} = \mathbf{F}^{\mathbf{T}}\mathbf{d}\mathbf{F}. \tag{48}$$

Expression (48) justifies the name *material rate of the deformation tensor* given to $\mathbf{D}$, since $\mathbf{d} = \boldsymbol{\chi} \circ \mathbf{D}$, $\mathbf{D} = \boldsymbol{\chi}^{-1} \circ \mathbf{d}$.

In addition, introducing (18) and (19) into (48) leads to

$$\mathbf{d} = \boldsymbol{\chi} \circ \left(\frac{\partial}{\partial t}(\boldsymbol{\chi}^{-1} \circ \mathbf{e}) \right) = \mathcal{L}_{\boldsymbol{\chi}}[\mathbf{e}]. \tag{49}$$

Hence, the rate of deformation tensor is the push-forward of the time derivative of the pull-back of the Almansi tensor. This operation is known as the *Lie derivative* of a tensor over the mapping $\boldsymbol{\chi}$ and it will be used in the following sections, being immediate that $\mathbf{d}$ is the Lie derivative of the Almansi tensor $\mathbf{e}$.

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Biographical Sketches

Manuel Doblaré was born in Cordoba (Spain) in July, 1956. He got the degree of Mechanical and Electrical Engineering at the University of Seville (Spain) in 1978 and PhD degree at the Polytechnique University in Madrid (Spain) in 1981. From 1978 to 1982 he was research assistant, when he got the position of assistant professor of Structural Mechanics at the PUM. In 1984, he was appointed as full professor at the Department of Mechanical Engineering of the University of Zaragoza (Spain) where he still teaches. He has occupied the positions of head of the Dept. of Mechanical Engineering (1985-87), dean of the Faculty of Engineering (1993-96) and Director of the Aragon Institute for Engineering Research (2002-07). Currently, he is head of the research group on Structural Mechanics and Materials Modeling (GEMM), and Scientific Director of the National Networking Center on Bioengineering, Biomaterials and Nanomedicine (CIBER-BBN). Prof. Doblaré has been elected as ordinary member of the Spanish Royal Academy of Engineering, the Royal Academy of Mathematics, Physics, Chemistry and Natural Sciences of Zaragoza and the World Council of Biomechanics, and awarded with several distinctions including the Aragon Prize for excellence in research or the Doctorate "Honoris Causa" at the Technical University of Cluj-Napoca (Romania). He was visiting scholar at the Universities of Southampton (Dept. of Civil Engineering-1981) and New York (Courant Institute of Mathematical Sciences-1983) and visiting professor at Stanford University (Division of Applied Mechanics-1990). He is member of different national and international scientific associations and of the editorial boards of several high impact journals where he has published more than 150 papers. He has given plenary, semiplenary and invited lectures in many international congresses and research fori, being internationally recognized in the field of Biomechanics. Prof. Doblaré's research interests are in computational solid mechanics with applications in structural integrity, biomechanics and mechanobiology, with special emphasis in hard and soft tissues modeling, interface behavior and interaction tissue-biomaterial, mechanobiological processes like bone remodeling, bone and wound healing, bone osteointegration or morphogenesis and, finally, tissue engineering.

Estefanía Peña became Associate Professor of the Department of Mechanical Engineering of the University of Zaragoza (Spain) in 2008. From 2001 to 2004 she was lecturer and from 2004 to 2008 Assistant Professor of Structural Mechanics at the UZ. She got the degree of Mechanical Engineering at the University of Zaragoza (Spain) in 2000. She achieved his Ph.D. in Computational Mechanics at the University of Zaragoza in 2004 and spent a post-doctoral stay at the Universities of Southampton in U.K. in 2004 and Joseph Fourier of Grenoble in France in 2005. She is member of Group of Structural Mechanics and Material Modeling of the I3A (Aragón Institute of Engineering Research). Her current research is related to Biomechanics, mainly in the field of Mechanics of Soft Tissues –as blood vessels, muscle, ligament and tendons– mechanical behaviour of biomaterials and prostheses for clinical applications and experimental methods to characterize biological tissues.

José F Rodriguez received his undergraduate degree in Mechanical Engineering from the Universidad Simón Bolívar in Venezuela and his PhD from the University of Notre Dame in 1999. He worked at the Universidad Simón Bolívar from 1999 to 2003, in 2003 he moved to the Universidad de Zaragoza as a

Ramón y Cajal researcher. He is a an associate researcher since 2008. He has conducted research in developing theoretical and computational models of complex nonlinear materials and systems with applications to living tissues and rubber-like solids. Another are of his research is computer electrophysiology and cell biophysics where he studies the electrical activity of the heart and the electro-mechanical interaction in the heart muscle and cardiac cells.