NONLINEAR ELASTICITY AND ITS ROLE IN CONTINUUM THEORIES

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Summary

Nonlinear elasticity involves a number of concepts and methods that also occur in other theories within continuum mechanics. An overview of nonlinear elasticity is first presented that discusses these concepts and methods. Two specific additional theories are then discussed, nonlinear viscoelasticity and mixture theory. It is shown how these theories utilize generalizations of the concepts and methods introduced in the context of nonlinear elasticity. The intention is to show that if one is educated in nonlinear elasticity, one is also prepared to work in nonlinear viscoelasticity and mixture theory.

1. Introduction

In the period from the 1950's through the 1970's, nonlinear elasticity was the subject of a great deal of interest and experienced a rapid growth. Many workers explored the concepts involved in its formulation and the physical and mathematical consequences of its theoretical structure. With the insights and experience gained from studying nonlinear elasticity, prominent researchers such as R. S. Rivlin, A. E. Green, J. E. Adkins and A. J. M. Spencer worked on the formulation of several additional theories in continuum mechanics (*An interesting overview of the activity in this period can be gained by reading the titles of the papers listed in the collected works of Rivlin (1996).*). Among these are the formulation of constitutive equations for nonlinear phenomena, nonlinear viscoelasticity, and the theory of interacting continua. Many topics within nonlinear elasticity provided the foundation for their work. In particular, the kinematics of large deformations, the use of polar decomposition, material symmetry considerations and the theory of invariants, polynomial matrix equations and the Cayley-Hamilton theorem were developed further in order to be of use in these later theories. In addition, it was found that many fascinating results in nonlinear elastic materials had their counterparts in these theories.

Thus, with an understanding of nonlinear elasticity, one is well positioned to understand the formulation of additional theories and to then explore them. It is the intent of this article to show the reader that an education in nonlinear elasticity means that one is also educated, though perhaps unaware of it, in the foundations of several additional subjects. In this article, we first present a summary of nonlinear elasticity. We then present two additional subjects in continuum mechanics, nonlinear viscoelasticity and mixture theory, emphasizing the role played by nonlinear elasticity in their formulation

and application.

2. Nonlinear Elasticity

This section contains an outline of the topics of nonlinear elasticity that are relevant to the purposes of this article. For more detailed presentations of the subject, see the standard reference by Ogden (1984) and the review article by Beatty (1987).

2.1. Kinematics

A body is a set of material points called particles. A typical particle P is identified or labeled by its position vector $\mathbf{X}$ at some reference time t_o. The domain of $\mathbf{X}$ at time t_0 is called a reference configuration of the body. Let $\mathbf{x}$ denote the position of particle P at time t. The motion of particle P is described by the vector function

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t). \tag{1}$$

For a fixed $\mathbf{X}$, (1) gives the path of particle P as time t increases. At a fixed time t, (1) gives the positions $\mathbf{x}$ of all particles of the body. The domain of $\mathbf{x}$ at time t is called the current configuration of the body. This motion is assumed to be one-to-one so that (1) can be inverted to express the label of a particle in terms of its current position

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t) \tag{2}$$

The velocity and acceleration of particle P are given by

$$\dot{\mathbf{x}}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t} \tag{3}$$

$$\ddot{\mathbf{x}}(\mathbf{X}, t) = \frac{\partial^2 \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2} \tag{4}$$

The superposed dot denotes the derivative with respect to time holding the particle label fixed. When the independent spatial variable is $\mathbf{X}$, (3) and (4) give the material (or Lagrangian) description of the velocity and acceleration. Relation (2) can be used to change the independent spatial variable in (3) and (4) from $\mathbf{X}$ to $\mathbf{x}$ giving

$$\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}}\left(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t\right) \tag{5}$$

$$\mathbf{a}(\mathbf{x}, t) = \ddot{\mathbf{x}}\left(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t\right) \tag{6}$$

Relations (5) and (6) give the spatial (or Eulerian) description of the velocity and acceleration.

In the remainder of this sub-section, explicit mention of the arguments is omitted for

ease of presentation. In order that the motion be one-to-one, it is assumed that

$$J = \det \mathbf{F} > 0, \tag{7}$$

where

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \tag{8}$$

is called the deformation gradient. This tensor contains information that compares the rotation and distortion in the neighborhood of a material particle at time t to its neighborhood in the reference configuration. The quantity J defined in (7) represents the ratio of the current volume of a neighborhood of a particle to its volume in the reference configuration. Application of the Polar Decomposition Theorem of linear algebra leads to

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{9}$$

in which the factors $\mathbf{U}, \mathbf{V}$, and $\mathbf{R}$ satisfy

$$\mathbf{RR}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}, \tag{10}$$

$$\mathbf{U} = \mathbf{U}^{\mathrm{T}}, \quad \mathbf{V} = \mathbf{V}^{\mathrm{T}}. \tag{11}$$

The orthogonal tensor $\mathbf{R}$ represents the rigid body rotation of the neighborhood of the particle while $\mathbf{U}$ and $\mathbf{V}$, called the right and left stretch tensors, describe the local deformation of the neighborhood. It is tedious to compute tensors $\mathbf{R}$ and $\mathbf{U}$ from the deformation gradient $\mathbf{F}$. For this reason, one defines the more easily computed tensor,

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2. \tag{12}$$

The tensor $\mathbf{C}$, called the right Cauchy-Green strain tensor, has the same principle directions as $\mathbf{U}$ and its principal values are the squares of those of $\mathbf{U}$. Hence, $\mathbf{C}$ is regarded as containing the same information as $\mathbf{U}$ about the local deformation of the neighborhood. The deformation gradient $\mathbf{F}$, its polar decomposition and the tensor $\mathbf{C}$ play a fundamental role in the development of more advanced theories in continuum mechanics.

2.2. Field Equations

The field equations are stated in local form. See Ogden (1984) for a more complete development.

2.2.1. Conservation of Mass

Let ρ_0 and ρ denote the mass per unit volume at a particle in the reference and current configurations, respectively. The statement of conservation of mass is that at each

particle

$$\rho J = \rho_0 \tag{13}$$

2.2.2. Conservation of Linear and Angular Momentum

The body force per unit mass on a material particle in the current configuration is denoted by $\mathbf{b}$, the unit outer normal to an area element on the surface of the current configuration is denoted by $\mathbf{n}$, the force per unit area on this surface area element is denoted by $\mathbf{T}$ and the Cauchy or true stress tensor is denoted by $\boldsymbol{\sigma}$. Application of the Principles of the Conservation of Linear Momentum in the current configuration leads to

$$\mathbf{T} = \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n} \tag{14}$$

on the surface and to

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a} \tag{15}$$

at each point within the current configuration. The Principle of the Conservation of Angular Momentum leads to the statement that the Cauchy stress tensor is symmetric,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \tag{16}$$

2.2.3. Conservation of Energy

The internal energy per unit mass at a material particle is denoted by 'e', the rate of heat supply per unit mass to a particle is denoted by 'r', the heat energy per unit time (heat flux) per unit area through a surface area element of the current configuration with unit outer normal $\mathbf{n}$ is denoted by 'q', and the heat flux vector is denoted by $\mathbf{q}$. Application of the Principle of the Conservation of Energy in the current configuration leads to

$$q = \mathbf{q}^{\mathrm{T}}\mathbf{n} \tag{17}$$

on the surface and to

$$\rho\dot{e} = \operatorname{tr}\boldsymbol{\sigma}\mathbf{F}\mathbf{F}^{-1} + \rho r - \operatorname{div}\mathbf{q} \tag{18}$$

at each point within the current configuration.

2.2.4. Entropy Inequality

The absolute temperature is denoted by θ and the entropy per unit mass of a particle is denoted by η. It is assumed that the entropy satisfies Clausius-Duhem inequality, whose local form is

$$\rho\dot{\eta} \geq \frac{\rho r}{\theta} - \mathrm{div}\left(\frac{\mathbf{q}}{\theta}\right). \tag{19}$$

It is convenient for the purpose of presenting the constitutive theory to introduce the Helmholtz free energy

$$\psi = e - \eta\theta \ . \tag{20}$$

An alternate version of (19), obtained with the use of (18) and (20), is

$$-\rho\left(\dot{\psi} + \eta\dot{\theta}\right) + \mathrm{tr}\boldsymbol{\sigma}\mathbf{F}\mathbf{F}^{-1} \geq \frac{1}{\theta}\mathbf{q}^{\mathrm{T}} \cdot \mathrm{grad}\,\theta \ . \tag{21}$$

Field equations (15) and (18) hold in the current configuration, while boundary conditions (14) and (17) hold on the surface of the current configuration. This is inconvenient because the current configuration is usually unknown and is determined as part of the process of solving a particular problem. For this reason, these equations are transformed so that they hold in the known reference configuration. The results of this transformation are presented here. For a detailed derivation, see Ogden (1984).

Let $\mathbf{N}$ denote the unit outer normal to a surface area element in the reference configuration whose unit outer normal is $\mathbf{n}$ in the current configuration. The quantity

$$\boldsymbol{\Pi} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} \tag{22}$$

is known as the first Piola-Kirchhoff stress tensor and represents a force per unit area in the reference configuration. Boundary condition (14) becomes

$$\mathbf{T}^{\mathrm{O}} = \boldsymbol{\Pi}^{\mathrm{T}}\mathbf{N}, \tag{23}$$

Where $\mathbf{T}^{\mathrm{O}}$ is the surface force per unit reference area. The equation of linear momentum (15) becomes

$$\mathrm{Div}\,\boldsymbol{\Pi} + \rho_0\mathbf{b} = \rho_0\ddot{\mathbf{x}} \ . \tag{24}$$

(Div denotes the divergence operator with respect to $\mathbf{X}$.) Transformation of the equation for the conservation of energy leads to the modified heat flux vector,

$$\mathbf{Q} = J\mathbf{q}\mathbf{F}^{-\mathrm{T}}, \tag{25}$$

where vector $\mathbf{Q}$ represents heat flux per unit reference area. Boundary condition (17) becomes

$$Q = \mathbf{Q}^{\mathrm{T}}\mathbf{N}, \tag{26}$$

in which Q is the heat flux per unit area of the surface in the reference configuration. The conservation of energy equation (18) becomes

$$\rho_o \dot{e} = \mathrm{tr}\mathbf{\Pi}\dot{\mathbf{F}}\mathbf{F}^{-1} + \rho_o r - \mathrm{div}\mathbf{Q}\,. \tag{27}$$

2.3. Constitutive Equations for Nonlinear Elastic Response

An elastic solid, in the presence of heat conduction, is defined by the assumption that the stress $\boldsymbol{\sigma}$, internal energy 'e', specific entropy η, and by (20), the Helmholtz free energy ψ at time t depend only on the deformation gradient $\mathbf{F}$, temperature θ and temperature gradient $\mathrm{grad}\,\theta$ at time t. Thermodynamic arguments (see Ogden (1984)) show that the stress, internal energy, specific entropy and Helmholtz free energy do not depend on the temperature gradient $\mathrm{grad}\,\theta$. Hence, the constitutive equation for the stress has the form

$$\boldsymbol{\sigma} = \mathcal{F}\left[\mathbf{F}(t), \theta(t)\right], \tag{28}$$

in which $\mathcal{F}$ is a tensor valued response function. There are three main sources of restrictions on $\mathcal{F}$: (a) the influence of superposed rigid body motions, (b) material symmetry, (c) restrictions due to thermodynamics.

2.3.1. Influence of superposed rigid body motions

Consider the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ in (1). Suppose that the body undergoes a second motion $\mathbf{x} = \boldsymbol{\chi}^*(\mathbf{X}, t)$ obtained from the first by a superposed rigid body motion,

$$\boldsymbol{\chi}^*(\mathbf{X}, t) = \mathbf{Q}(t)\left[\boldsymbol{\chi}(\mathbf{X}, t) - \mathbf{d}(t)\right]. \tag{29}$$

Vector $\mathbf{d}(t)$ represents a rigid body translation. $\mathbf{Q}(t)$ represents a rigid body rotation and satisfies

$$\mathbf{Q}(t)\mathbf{Q}(t)^{\mathrm{T}} = \mathbf{Q}(t)^{\mathrm{T}}\mathbf{Q}(t) = \mathbf{I}\,. \tag{30}$$

It is assumed that the superposed rigid body motion only affects the stress by rotating it. This leads to the condition that

$$\mathcal{F}\left[\mathbf{Q}(t)\mathbf{F}(t), \theta(t)\right] = \mathbf{Q}(t)\mathcal{F}\left[\mathbf{F}(t), \theta(t)\right]\mathbf{Q}(t)^{\mathrm{T}} \tag{31}$$

for any $\mathbf{Q}(t)$ satisfying (30). This, when combined with (9) leads to the statement that (28) is of the form

$$\boldsymbol{\sigma} = \mathbf{R}(t)\,\mathcal{F}\left[\mathbf{U}(t), \theta(t)\right]\mathbf{R}(t)^{\mathrm{T}}\,. \tag{32}$$

Because (i) the determination of $\mathbf{R}$ and $\mathbf{U}$ from $\mathbf{F}$ using (9) is tedious, (ii) $\mathbf{U}$ and $\mathbf{C}$

contain essentially the same information about the local deformation and (iii) $\mathcal{F}$ is as yet arbitrary, (32) is usually restated without loss in generality as,

$$\boldsymbol{\sigma} = \mathbf{F}(t)\mathcal{G}\left[\mathbf{C}(t),\theta(t)\right]\mathbf{F}(t)^{\mathrm{T}}, \tag{33}$$

in which $\mathcal{G}$ is a new response function. It is straightforward to show that (33) satisfies (31).

2.3.2. Material Symmetry

The concept of material symmetry arises from the fact that a material has some physical microstructure in its reference configuration, such as a crystal structure or randomly oriented macromolecules. Consider a sample of material in its reference configuration and its microstructure. Suppose there is a transformation of this reference configuration to a new configuration such that the material appears to have the same microstructure as before. Let both the original and transformed configurations be subjected to the same homogeneous deformation with deformation gradient **F**. The underlying microstructures, which appear to be the same in their respective reference configurations, are distorted in the same way. The stresses are assumed to be the same and these configurations are considered to be mechanically equivalent.

A transformation of the original reference configuration to one that is mechanically equivalent is a linear transformation is denoted by **H**. One restriction on **H** is that it produce no volume change and this leads to the condition that $\left|\det \mathbf{H}\right| = 1$. In addition, for most equivalent microstructures of interest, **H** is a rotation or a reflection and satisfies

$$\mathbf{H}\mathbf{H}^{\mathrm{T}} = \mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{I}\ . \tag{34}$$

Symmetries of a material are described by giving the set of transformations **H** that lead to equivalent microstructures. These form a mathematical entity called a material symmetry group.

For each transformation **H** of a material symmetry group, the above discussion implies that the constitutive equation (28) satisfies

$$\mathcal{F}\left[\mathbf{F}(t),\theta(t)\right] = \mathcal{F}\left[\mathbf{F}(t)\mathbf{H},\theta(t)\right] \tag{35}$$

Material symmetry restrictions can be imposed on the response function $\mathcal{G}$ by substituting (33) into (35),

$$\mathbf{H}^{\mathrm{T}}\mathcal{G}\left[\mathbf{C}(t),\theta(t)\right]\mathbf{H} = \mathcal{G}\left[\mathbf{H}^{\mathrm{T}}\mathbf{C}(t)\mathbf{H},\theta(t)\right] \tag{36}$$

2.3.3. Thermodynamical Considerations

Thermodynamical arguments utilizing the entropy inequality in the form (21), that are by now standard, lead to the result that the Helmholtz free energy is independent of the

temperature gradient,

$$\psi = \psi(\mathbf{F}, \theta) \tag{37}$$

and to the following relations

$$\boldsymbol{\sigma} = J^{-1} \frac{\partial(\rho_0 \psi)}{\partial \mathbf{F}} \mathbf{F}^{\mathrm{T}} \ , \tag{38}$$

$$\eta = -\frac{\partial \psi}{\partial \theta} \ , \tag{39}$$

$$\mathbf{q} \cdot \operatorname{grad} \theta \le 0 \ . \tag{40}$$

Since all physical variables are evaluated at time t, explicit dependence on time is omitted for notational convenience. These thermodynamical restrictions are now combined with the restrictions associated with the influence of superposed rigid body motion and of material symmetry. Since ψ is a scalar, the restriction imposed by the independence of a superposed rigid body motion requires that $\psi(\mathbf{F}, \theta)$ satisfy

$$\psi(\mathbf{F}, \theta) = \psi(\mathbf{QF}, \theta) . \tag{41}$$

This is satisfied by writing

$$\rho_0 \psi(\mathbf{F}, \theta) = W(\mathbf{C}, \theta) \ . \tag{42}$$

When this is substituted into (38), the expression for the stress reduces to

$$\boldsymbol{\sigma} = J^{-1} \mathbf{F} \left[\frac{\partial W}{\partial \mathbf{C}} + \left(\frac{\partial W}{\partial \mathbf{C}} \right)^{\mathrm{T}} \right] \mathbf{F}^{\mathrm{T}} . \tag{43}$$

Note that this is of the form (33).

Since $W(\mathbf{C}, \theta)$ is a scalar, the earlier discussion on material symmetry leads to the restriction

$$W(\mathbf{C}, \theta) = W(\mathbf{H}^T \mathbf{C} \mathbf{H}, \theta) \ . \tag{44}$$

2.3.4. Material Symmetry Restrictions-General Results

The method of determination of the form of $\mathcal{G}$ that satisfies (36) and of W that satisfies (44) has been presented in the classic review article by Spencer (1971). For each type of material symmetry, there is a set of basic scalar functions $I_k(\mathbf{C})$, $k = 1, \cdots, N$, called

invariants, that have the property,

$$I_k(\mathbf{C}) = I_k\left(\mathbf{H}^{\mathrm{T}}\mathbf{C}\mathbf{H}\right) \tag{45}$$

for each transformation $\mathbf{H}$ of the material symmetry group. W can be expressed as a function of these invariants

$$W(\mathbf{C},\theta) = \hat{W}\left(I_1(\mathbf{C}),\ldots,I_N(\mathbf{C}),\theta\right). \tag{46}$$

With W written as (46) and with property (45), it is seen that (44) is satisfied. Then (43) becomes

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\left\{\sum_{k=1}^{N}\frac{\partial W}{\partial I_k}\left[\frac{\partial I_k}{\partial \mathbf{C}} + \left(\frac{\partial I_k}{\partial \mathbf{C}}\right)^{\mathrm{T}}\right]\right\}\mathbf{F}^{\mathrm{T}}. \tag{47}$$

This is of the form

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\left\{\sum_{k=1}^{N}\Phi_k\mathbf{P}_k\right\}\mathbf{F}^{\mathrm{T}}, \tag{48}$$

in which $\Phi_k\left(I_1,\ldots,I_N\right)$ are scalars to be determined for a particular material and

$$\mathbf{P}_k = \frac{\partial I_k}{\partial \mathbf{C}} + \left(\frac{\partial I_k}{\partial \mathbf{C}}\right)^{\mathrm{T}} \tag{49}$$

are matrix polynomials associated with the particular material symmetry under consideration.

It should be noted that (47) arose out of the restrictions due to thermodynamics, the influence of superposed rigid body motion and material symmetry. One of the results presented in Spencer (1971) is that if a function $\mathcal{G}$ satisfies condition (36), it can be written in the form

$$\mathcal{G} = \sum_{k=1}^{N}\tilde{\Phi}_k\mathbf{P}_k\ , \tag{50}$$

in which $\tilde{\Phi}_k$ are scalar functions of the invariants and $\mathbf{P}_k$ is defined in (49). The consequence of the thermodynamical restriction is that the scalar coefficients $\tilde{\Phi}_k$ can be expressed in terms of a single quantity, the Helmholtz free energy. It has been shown, (see Spencer (1971)), that if a tensor valued function of a set of vector and tensor arguments satisfies a material symmetry condition analogous to (36), it can be expressed in a form similar to (50), i.e. as a sum of terms, each being a product of a scalar and a tensor. The scalar coefficient is a function of basic scalar invariants and the

tensor is a basic tensor function, both being associated with the particular material symmetry and set of independent variables under consideration.

2.3.5. Material Symmetry Restrictions – Specific Symmetries

The symmetries commonly used to describe nonlinear elastic materials are isotropy, transverse isotropy and orthotropy.

ISOTROPY: A material is said to have hemihedral or proper isotropy if the set of material symmetry transformations $\mathbf{H}$ consists of rotations. It is said to have holohedral or full isotropy if the group of material symmetry transformations consists of rotations and a central reflection. In either case, the set of strain invariants is

$$I_1(\mathbf{C}) = \mathrm{tr}\mathbf{C},\ I_2(\mathbf{C}) = \frac{1}{2}\left[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)\right],\ I_3(\mathbf{C}) = \det\mathbf{C}\,, \tag{51}$$

TRANSVERSE ISOTROPY: Let the axes of a Cartesian coordinate system define directions with respect to the material in the reference configuration. A material is said to have transverse isotropy with respect to the direction indicated by, say, the X_3 axis, if the set of material symmetry transformations consists of rotations about this axis. Different classes of transverse isotropy arise by also including reflections about or perpendicular to this axis. In any case, the set of strain invariants consists of the invariants in (51) and also

$$I_4(\mathbf{C}) = C_{33},\ I_5(\mathbf{C}) = C_{13}{}^2 + C_{23}{}^2. \tag{52}$$

ORTHOTROPY: Let the axes of Cartesian coordinate system define directions with respect to the material in the reference configuration. A material is said to have orthotropy if the set of material symmetry transformations consists of 90^o rotations about each of the axes. Different classes of orthotropy arise by also including reflections about or perpendicular to these axes. The set of invariants consists of

$$I_1(\mathbf{C}) = C_{11},\ I_2(\mathbf{C}) = C_{22},\ I_3(\mathbf{C}) = C_{33},\ I_4(\mathbf{C}) = C_{12}{}^2,\ I_5(\mathbf{C}) = C_{23}{}^2,\ I_6(\mathbf{C}) = C_{31}{}^2 \tag{53}$$

2.3.6. Material Symmetry Restrictions – Forms For Constitutive Equations

For an isotropic material, use of (51) in (47) leads to

$$\boldsymbol{\sigma} = 2J^{-1}\mathbf{F}\left[\left(\frac{\partial\hat{W}}{\partial I_1} + I_1\frac{\partial\hat{W}}{\partial I_2} + I_2\frac{\partial\hat{W}}{\partial I_3}\right)\mathbf{I} - \left(\frac{\partial\hat{W}}{\partial I_2} + I_1\frac{\partial\hat{W}}{\partial I_3}\right)\mathbf{C} + \frac{\partial\hat{W}}{\partial I_3}\mathbf{C}^2\right]\mathbf{F}^{\mathrm{T}}. \tag{54}$$

The calculation of $\mathbf{FIF}^{\mathrm{T}}$, $\mathbf{FCF}^{\mathrm{T}}$ and $\mathbf{FC}^2\mathbf{F}^{\mathrm{T}}$ in (54), in view of (12), induces a new strain tensor $\mathbf{B}$,

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}\ , \tag{55}$$

called the left Cauchy-Green strain tensor. In addition, it is readily shown for the invariants in (51) that $I_\alpha(\mathbf{C}) = I_\alpha(\mathbf{B})$. Then (54) becomes

$$\boldsymbol{\sigma} = 2J^{-1}\left[I_3 \frac{\partial \hat{W}}{\partial I_3}\mathbf{I} + \left(\frac{\partial \hat{W}}{\partial I_1} + I_1 \frac{\partial \hat{W}}{\partial I_2} \right)\mathbf{B} - \frac{\partial \hat{W}}{\partial I_2}\mathbf{B}^2 \right], \tag{56}$$

where $\partial \hat{W} / \partial I_\alpha$ is now a function of $I_\alpha(\mathbf{B})$. In deriving (56), use has been made of the Cayley-Hamilton theorem

$$\mathbf{B}^3 = I_1\mathbf{B}^2 - I_2\mathbf{B} + I_3\mathbf{I} \ . \tag{57}$$

An alternate form of the constitutive equation is obtained if (57) is multiplied by $\mathbf{B}^{-1}$ and the resulting expression for $\mathbf{B}^2$ is substituted into (56),

$$\boldsymbol{\sigma} = 2J^{-1}\left[\left(I_2 \frac{\partial W}{\partial I_2} + I_3 \frac{\partial W}{\partial I_3} \right)\mathbf{I} + \frac{\partial W}{\partial I_1}\mathbf{B} - I_3 \frac{\partial W}{\partial I_2}\mathbf{B}^{-1} \right]. \tag{58}$$

Relations analogous to (54) can be developed for transversely isotropic and orthotropic materials. For the sake of brevity of presentation, this will not be done here.

Two comments are important at this point. First, the Cayley-Hamilton theorem has been indispensable in the development of representations of constitutive equations for different physical phenomena and material symmetries. Second, it is possible to eliminate **F** and **C** from (54) and introduce the strain tensor **B** only for an isotropic elastic material.

2.4. Constraints

The possible motions of a body may be limited by constraints such as incompressibility or inextensibility in certain directions. Such constraints impose restrictions on the constitutive equations. Discussion here will be restricted to the constraint of incompressibility.

In many polymeric materials, the volume change during deformation is observed to be very small. This leads to an idealized material model for which any possible motion must satisfy the constraint,

$$J = \det\mathbf{F} = 1 \ . \tag{59}$$

Motions that satisfy (59) are described as being isochoric. The constraint condition (59) implies that

$$I_3 = \det\mathbf{B} = J^2 = 1 \ . \tag{60}$$

Reconsideration of the restrictions of the thermodynamics of materials with constraint (59) leads to a modified form for (38),

$$\boldsymbol{\sigma} = -p\mathbf{I} + \rho_o \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^{\mathrm{T}} \ , \tag{61}$$

in which p is an arbitrary scalar. Note, by (13) and (59) that $\rho = \rho_0$. The restriction imposed by consideration of the influence of superposed rigid body motions must still be satisfied so that (42) again holds. When this is used in (61), the result is a modified form of (43),

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{F}\left[\frac{\partial W}{\partial \mathbf{C}} + \left(\frac{\partial W}{\partial \mathbf{C}}\right)^{\mathrm{T}}\right]\mathbf{F}^{\mathrm{T}} . \tag{62}$$

If the material is isotropic in its reference configuration, material symmetry considerations (46), (51) and (60) lead to

$$W(\mathbf{C}, \theta) = \tilde{W}(I_1, I_2, \theta) \ . \tag{63}$$

When this is substituted into (62), the calculations and use of the Cayley-Hamilton theorem lead to results analogous to (56) and (58),

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\left[\left(\frac{\partial \tilde{W}}{\partial I_1} + I_1 \frac{\partial \tilde{W}}{\partial I_2}\right)\mathbf{B} - \frac{\partial \tilde{W}}{\partial I_2}\mathbf{B}^2\right] . \tag{64}$$

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\left[\frac{\partial \tilde{W}}{\partial I_1}\mathbf{B} - \frac{\partial \tilde{W}}{\partial I_2}\mathbf{B}^{-1}\right] . \tag{65}$$

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