BASIC MECHANICS OF MATERIALS

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Keywords: stress, strain, displacements, equilibrium of forces, stress-strain relations, elasticity, principal stresses, principal directions, plane stress, plane strain, stress function, bi-harmonic equation, compatibility, orthotropic materials

Contents

Summary

The Mechanics of Materials establishes a simplified analytical methodology based on linear and elastic relationships between loads acting on objects and their geometries in order to determine and analyze the state of stress at their critical points. The present article presents the main topics of the Mechanics of Materials concerned with those relationships.

1. Introduction

The Mechanics of Materials (as well as the Theory of Elasticity, Experimental Stress Analysis, and the Numerical Methods such as the Finite Element Method) has two objectives. First, it determines the stresses that act on the surfaces of volume elements that represent the critical points of loaded structures. Second, it establishes equivalent stresses based on strength criteria associated with possible failure modes to help in the designing and analyzing of the structures.

Defining the state of stress at a point of a loaded structure entails:

1. Knowing and establishing the loads acting on the structure
2. Determining reactions and constructing free body diagrams
3. Selecting critical areas and sections of the structural element
4. Determining critical points at the critical sections
5. Calculating stresses that act on planes that define the volume elements that model the critical points

Analyzing the state of stress involves:

1. Determining the principal stresses at the critical points
2. Knowing the possible failure modes that may develop from the critical points
3. Determining the material's strengths at each critical point based on each failure mode
4. Calculating equivalent stresses based on strength criteria applicable to each failure mode
5. Comparing the equivalent stresses with the material's strengths
6. Calculating safety factors and probability of failure at each critical point based on the equivalent stresses and the material's strength comparisons.

Structural loads are predicted and evaluated by means of:

1. Previous experience with analogous structures
2. A mathematical model associated with the interpretation of the physical phenomena
3. Experimental determination:
    - Using load cells
    - Using sensors such as strain gages, using displacement, velocity and accelerometer transducers, and associating their resulting measurements with mathematical models and calibration procedures.

In determining structural loads it is important to consider the following aspects:

1. The nature of the loading: if it is caused by a static, impact or vibratory phenomenon
2. The loading history and number of cycles
3. The temperature and its variation over time, its variation along the structure's dimensions and its influence on the material's properties
4. The existence of residual stresses caused by fabrication, mounting, welding, heat treatments and service overloads
5. The uncertainty associated with the procedures of load determination

2. The State of Stress at a Point of the Loaded Structure

Let P_i and M_i be the force and moment loadings acting on a deformable body that is in equilibrium. The body is sectioned by plane A at the point of interest. Plane A is defined by its normal vector $\mathbf{n}_A$, as illustrated in Figure 1. The equilibrium of the left part of the plane-sectioned body will be maintained if the resulting force and moment – R_A and M_A, respectively – of the loadings acting on the right part of the body are calculated and positioned to act on the sectioning plane.

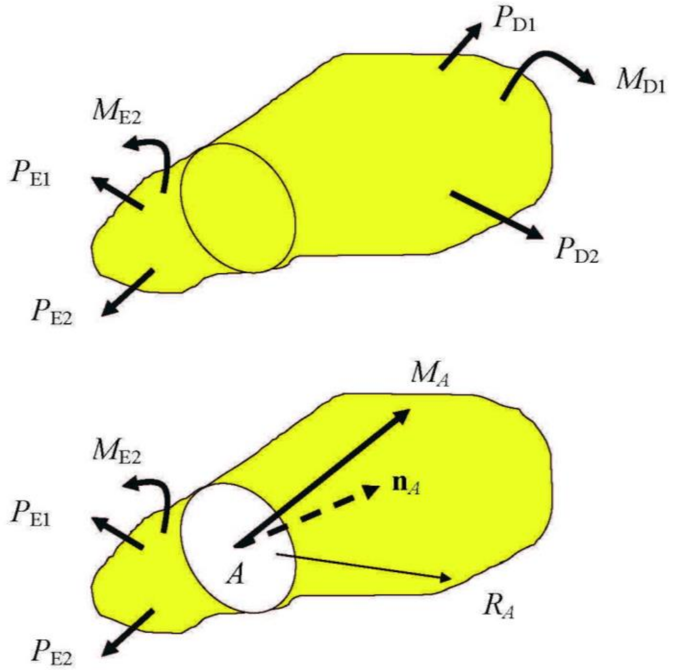

Figure 1. Deformable body subjected to external loading. R_A and M_A are the resultants of force and momentum loadings that act on the right part of the body.

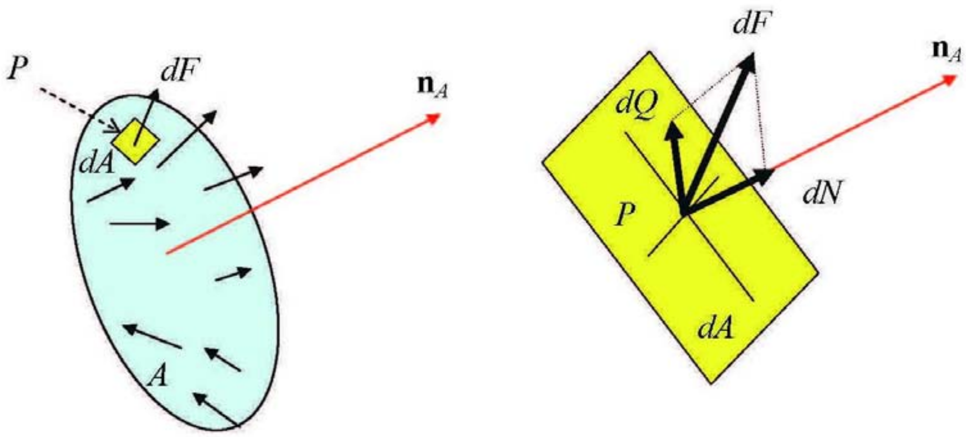

Figure 2. Total, normal and shear forces acting on the small area dA defined by plane A, which crosses the body in equilibrium at point P.

Loadings R_A and M_A result from the connecting forces dF that act at each point of sectioning plane A. In this case, the cross section defined by plane A is composed of small areas dA where connecting forces dF act. Each dF can be resolved into two components, as depicted in Figure 2: one, dN, is normal for the area dA; the other, dQ, is tangent to the area dA. Thus, total stress T, normal stress σ, and shear stress τ are defined by Eq. (1). These stresses act on area dA, which is defined by plane A, which, in turn, sections the body at point P.

$$\vec{T} = \frac{\overrightarrow{dF}}{dA} \qquad \vec{\sigma} = \frac{\overrightarrow{dN}}{dA} \qquad \vec{\tau} = \frac{\overrightarrow{dQ}}{dA} \tag{1}$$

Figure 3 shows a small volume of material defined by four different planes that cross the body at point P. The material's volume is represented by a small tetrahedron. The equilibrium of the tetrahedron must be maintained by the forces that act on the four areas and by the unit volume force *C*. This small force is caused by some mass originated force such as its own weight or magnetic attraction.

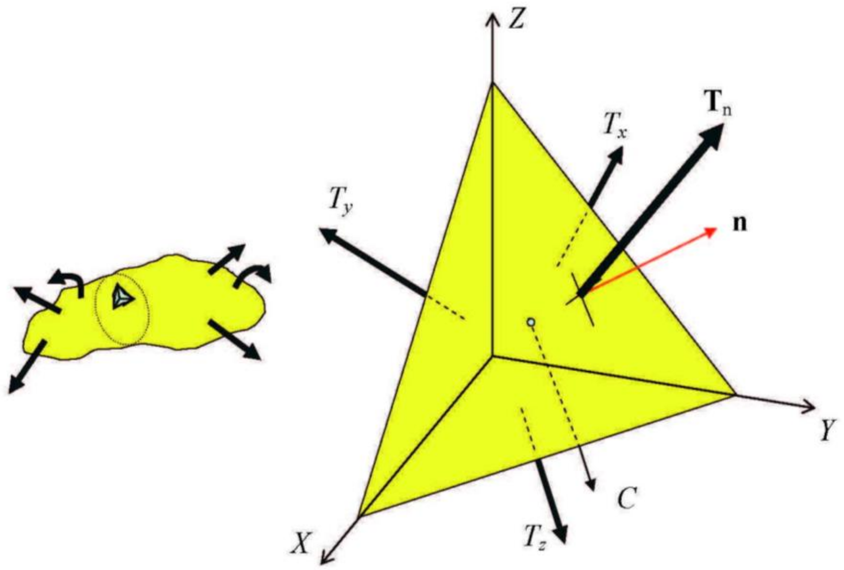

Figure 3. Tetrahedron formed by four planes that cross the body at point *P*

The equilibrium of the tetrahedron is given by Eq. (2). The forces acting on each face are calculated by the respective total stresses multiplied by the respective areas. Resolving the total stresses along the three coordinate axes *x*, *y* and *z*, as exemplified in Figure 3 and Eqs. (3) for the total stress T_z, yields Eq. (4).

$$\overrightarrow{T_x}.A_x + \overrightarrow{T_y}.A_y + \overrightarrow{T_z}.A_z + \vec{C}.V = \overrightarrow{T_n}.A_n \tag{2}$$

$$\left.\begin{array}{l} \vec{z} = 0.\vec{x} + 0.\vec{y} + 1.\vec{z} \\ \vec{n} = n_x.\vec{x} + n_y.\vec{y} + n_z.\vec{z} \\ \overrightarrow{T_n} = T_{nx}.\vec{x} + T_{ny}.\vec{y} + T_{nz}.\vec{z} \\ \overrightarrow{T_z} = \tau_{zx}.\vec{x} + \tau_{zy}.\vec{y} + \sigma_z.\vec{z} \end{array}\right| \rightarrow \left| \begin{array}{l} z = (0,0,1) \\ n = (n_x, n_y, n_z) \\ T_n = (T_{nx}, T_{ny}., T_{nz}) \\ T_z = (\tau_{zx}, \tau_{zy}, \sigma_z) \end{array}\right. \tag{3}$$

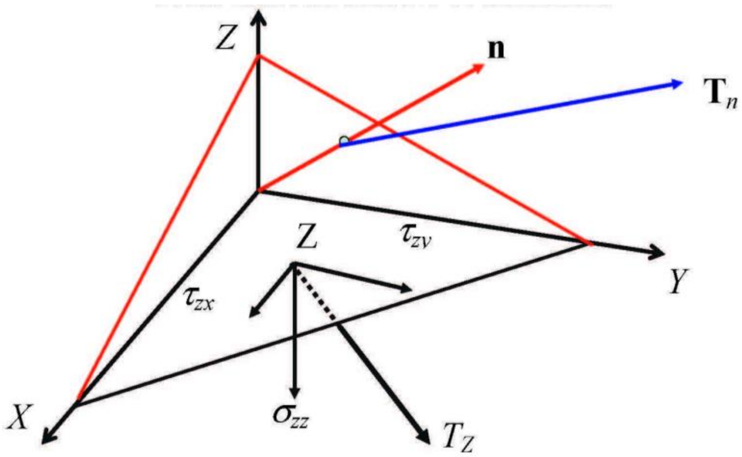

Figure 4. Decomposition of total stress T_z in three stresses according to directions X, Y and Z. The normal stress is called σ_{zz} or σ_z, Z being the outward normal that defines plane Z. Shear stresses acting on plane Z are denominated τ_z. The second sub-index defines the direction to which the shear stress is parallel.

Areas A_x, A_y and A_z can be calculated as a function of area A_n by means of scalar products, as depicted in Figure 5 and Eq. (4).

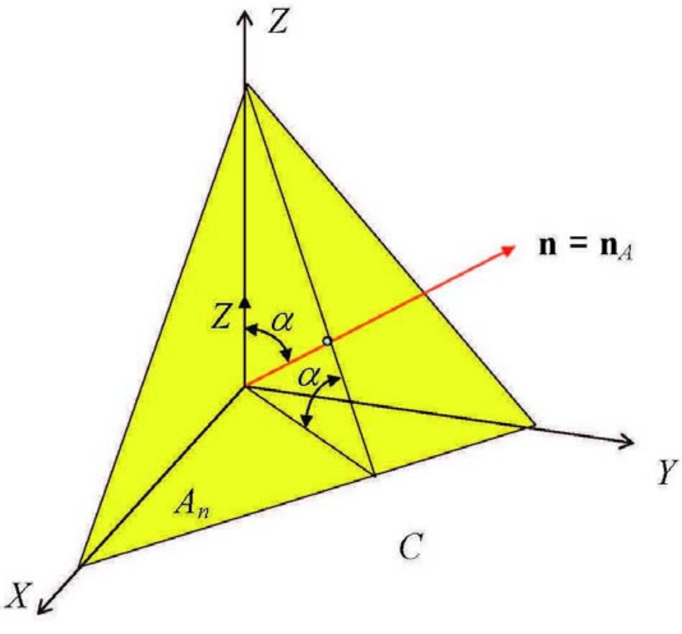

Figure 5. Calculation of area A_z by means of the projection of area A_n in plane Z

$$z=(0,0,1), \qquad n=\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \quad \rightarrow \quad A_z = A_n.\cos\alpha = A_n\left(\vec{z}.\vec{n}\right) = A_n.n_z \tag{4}$$

Equation (5) results from the equilibrium of forces and from the assumption that the body's force CV is small if compared with forces generated by the total stresses and may be disregarded.

$$
\begin{aligned}
&T_x.A_n.n_x + T_y.A_n.n_y + T_z.A_n.n_z + (C.V \to 0) = T_n.A_n \\
&\therefore T_x.n_x + T_y.n_y + T_z.n_z = T_n
\end{aligned}
\tag{5}
$$

Resolving Eq. (5) along directions *X*, *Y* and *Z* yields Eqs. (6), (7) and (8).

$$
\begin{aligned}
&\left(\sigma_x\dot{x} + \tau_{xy}\dot{y} + \tau_{xz}\dot{z}\right)n_x + \left(\tau_{yx}\dot{x} + \sigma_y\dot{y} + \tau_{yz}\dot{z}\right)n_y + \left(\tau_{zx}\dot{x} + \tau_{zy}\dot{y} + \sigma_z\dot{z}\right)n_z \\
&= \left(T_{nx}\dot{x} + T_{ny}\dot{y} + T_{nz}\dot{z}\right)
\end{aligned}
\tag{6}
$$

$$
\begin{cases}
\sigma_x.n_x + \tau_{yx}.n_y + \tau_{zx}.n_z = T_{nx} \\
\tau_{xy}.n_x + \sigma_y.n_y + \tau_{zy}.n_z = T_{ny} \\
\tau_{xz}.n_x + \tau_{yz}.n_y + \sigma_z.n_z = T_{nz}
\end{cases}
\tag{7}
$$

or

$$
\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} . \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} T_{nx} \\ T_{ny} \\ T_{nz} \end{bmatrix} \quad \text{or} \quad \boldsymbol{\sigma}\,\mathbf{n} = \mathbf{T}_n
\tag{8}
$$

Matrix $\boldsymbol{\sigma}$ is symmetrical due to equilibrium of moments around axes *X*, *Y* and *Z*. Thus, the identities expressed by Eqs. (9) hold.

$$
\tau_{xy} = \tau_{yx}, \quad \tau_{xz} = \tau_{zx}, \quad \tau_{yz} = \tau_{zy}
\tag{9}
$$

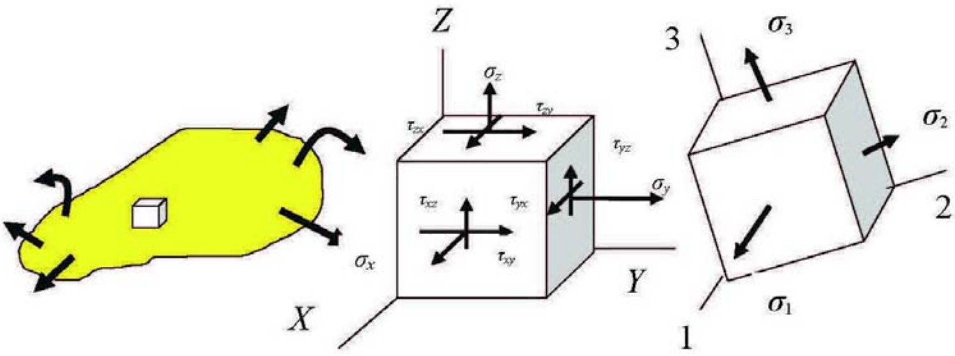

Figure 6. The state of stress at point P represented by two cubic elements, defined by planes and axes *X*, *Y* and *Z* and 1, 2 and 3. Planes and axes 1, 2 and 3 are respectively called the principal planes and principal directions

Matrix $\boldsymbol{\sigma}$ is a second order tensor defining the state of stress at point *P*. Using Eqs. (6), (7) or (8), the total stress $\mathbf{T}_n$ acting on any arbitrary plane *n* is completely defined with relation to axes *X*, *Y* and *Z*. In other words, the state of stress at a point is known if the stresses that act on the three planes that pass through the point are known. Note that these planes do not necessarily need to be orthogonal. Figure 6 shows the representation of the state of stress at point *P* by means of cubic elements defined by orthogonal planes *X*, *Y* and *Z* or planes 1, 2 and 3 passing through point *P*. Directions and planes denominated as 1, 2 and 3 are the so-called principal directions or principal planes. These are discussed in Section 3.

2.1. Prismatic Bar Subjected to Axial Loading

A simple example of applying the equilibrium equations in order to define the state of stress at points of a body is presented in this section. Figure 7 shows a prismatic body subjected to axial loading. The body is cut by a plane defined by its normal $\mathbf{n}_x$. It is reasonable to assume that: the total stresses acting on the small areas dA are equal; they coincide with the normal stresses; and that shear stresses are zero on these areas. Symmetry and equilibrium of momentum of cubic elements that belong to the external surfaces of the body help to justify the later assumption.

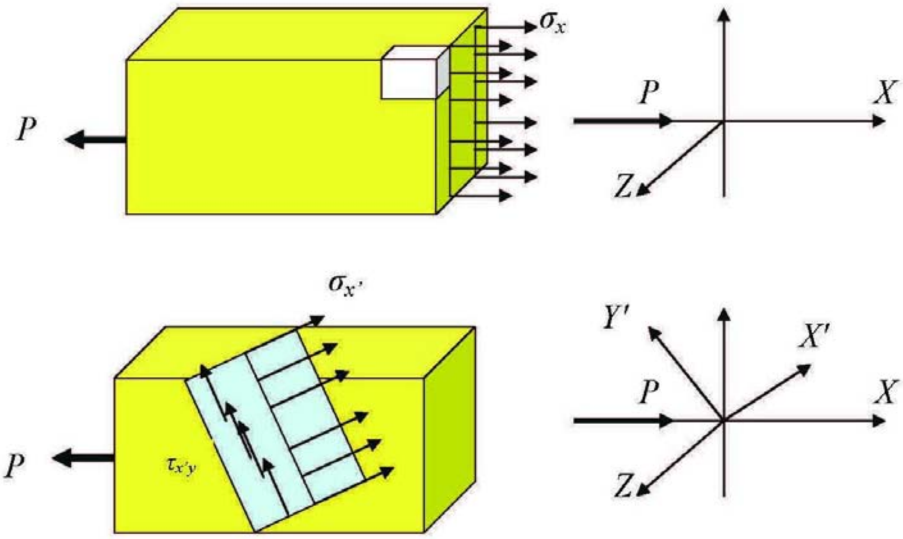

Figure 7. Virtual cutting planes to help define the state of stress at a point *P* of an axially loaded prismatic bar. The cutting planes are defined by their normals *X* and X', the latter forming an angle α with relation to the former.

The normal stresses at the points of cross section are named σ_x. Equilibrium of the left part of the cut body with respect to the forces actuating in the axial direction yields:

$$\sum F_x = 0 \Rightarrow P - \int_A \sigma_x.dA = 0 \Rightarrow P - \sigma_x.A = 0 \Rightarrow \sigma_x = \frac{P}{A}$$

If equilibrium conditions are again imposed for directions *Y* and *Z* with the help of passing two other orthogonal planes defined by vectors $\mathbf{n}_y$ and $\mathbf{n}_z$, it will be seen that the stresses σ_y and σ_z, respectively, acting on faces *Y* and *Z* of a cubic element that represents point P will be zero. Using the fact that $\tau_{xy} = \tau_{xz} = 0$, that $\tau_{xy} = \tau_{yx}$ and $\tau_{xz} = \tau_{zx}$, and that $\tau_{zy} = \tau_{yz} = 0$, it is possible to completely define the matrix of stress [σ] with relation to axes *X*, *Y*, *Z* such as is given by Eq. (10).

$$\begin{bmatrix} \sigma_x = \dfrac{P}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

The state of stress $\boldsymbol{\sigma}$ can be defined with relation to planes X', Y' and Z, shown in Figure 7. Stresses $\sigma_{x'}$ and $\tau_{x'y'}$ are determined by passing the cutting plane X', by assuming uniformity and by using the equilibrium equations applied along directions X' and Y':

$$\sum F_x = 0 \Rightarrow P.\cos\alpha - \int_A \sigma_{x'}.dA_{x'} = 0 \Rightarrow P.\cos\alpha - \sigma_{x'}.A.\frac{1}{\cos\alpha} = 0$$

$$\Rightarrow \sigma_{x'} = \frac{P}{A}.\cos^2\alpha = \frac{P}{2A}\left(1+\cos 2\alpha\right)$$

$$\sum F_{y'} = 0 \Rightarrow P.\sin\alpha - \int_A \tau_{x'y'}.dA_{x'} = 0 \Rightarrow P.\sin\alpha - \tau_{x'y'}.A.\frac{1}{\cos\alpha} = 0$$

$$\Rightarrow \tau_{x'y'} = \frac{P}{A}.\sin\alpha.\cos\alpha = \frac{P}{2A} w \sin 2\alpha$$

In the same way, cutting plane Y' and the equilibrium conditions yield:

$$\begin{aligned} \sigma_{y'} &= \frac{P}{A}.\sin^2\alpha = \frac{P}{2A}\left(1-\cos 2\alpha\right) \\ \tau_{y'x'} &= \frac{P}{A}.\sin\alpha.\cos\alpha = \frac{P}{2A}\sin 2\alpha \end{aligned} \tag{11}$$

The state of stress at point *P*, defined with relation to directions X', Y' and Z, is represented by Eq. (12).

$$\begin{bmatrix} \sigma_{x'} = \dfrac{P}{2A}\left(1+\cos 2\alpha\right) & \tau_{x'y'} = \dfrac{P}{2A}\sin 2\alpha & 0 \\ \tau_{y'x'} = \dfrac{P}{2A}\sin 2\alpha & \sigma_{y'} = \dfrac{P}{2A}\left(1-\cos 2\alpha\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

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Biographical Sketch

José L.F. Freire: B.S. (1972) and M.Sc. (1975) degrees in mechanical engineering from the Catholic University of Rio de Janeiro (PUC-Rio); Ph.D. (1979) in Engineering Mechanics from Iowa State University of Science and Technology (ISU); associate professor of Mechanical Engineering at PUC-Rio and chairman of the Structural Integrity Laboratory; member of the Society for Experimental Mechanics and is its past president; major areas of research: Experimental Stress Analysis, Pipeline Engineering and Structural Integrity of Equipment and Structures.