MATHEMATICAL STRUCTURES OF COMPLEXITY

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Summary

This article discusses a variety of mathematically-inspired approaches to the study of complexity and their application to the study and understanding of complex systems. These approaches have their origin in the study of limitative results in logic, mathematical linguistics, and the theory of computation, based upon the work of Godel, Turing, Chomsky, and Kolmogorov, among many others. The topics touched upon include computational complexity, Kolmogorov complexity, algorithmic complexity, Lempel-Ziv complexity, logical depth, Moore's generalized shifts, Rasmussen's non-simulatability, Crutchfield's epsilon machines, and Sulis machines.

1. Introduction

Throughout much of its history, science, in its quest for universal laws and principles, has focused its attention upon the simplest of systems -- usually one or two elements, idealized in form and function, governed by linear relationships, and isolated from the remainder of reality. This approach has proven to be spectacularly successful with the discovery of the laws of conservation of mass, energy, momentum, the fundamental forces of gravity, electromagnetism, strong nuclear and weak nuclear together with their (almost complete) unification, the fundamental particles of nature, the periodic table of elements, the basic laws of chemical interaction, speciation, the genome. Science has been guided by a deep faith in the principle of reductionism, the belief that a whole is just the sum of its parts, and that the behavior and properties of the whole can be predicted or recreated from a knowledge of the behavior of its parts and of the interactions among these parts. Nature, in its full majesty, was thought to be merely complicated, consisting perhaps of legions of fundamental constituents interacting in myriad ways, but nevertheless still within the scope of theory to be described and predicted. Nature was governed by context-independent, objective, *deterministic* laws, the laws of machines, with events following one another in a predictable, lock step manner independent of any process by which these events might be observed. Einstein espoused this belief in his famous dictum, "God does not play dice". All that will be already exists implicitly within all that was.

The discoveries of the twentieth century may have significantly shaken our faith in *reductionism*, *objectivism* and *determinism*, but we still believe that the essential form of Nature's design lies within our grasp. Mathematics and physics have provided us with a nearly complete, qualitative theory of simple systems. Even though we may not know precisely what a particular simple system will do, we can provide a general description of what it may do, much as a story describes the general actions of its characters without becoming burdened in unnecessary detail. Simple stable systems, whose individual states can be represented as elements within some abstract *phase space (*describing such characteristics as position, momentum, angular momentum, energy and so forth) exhibit rather simple motions. They may remain indefinitely in a single state, termed *fixed point* behavior. They may travel along a fixed path over and over again, termed *periodic* or *limit cycle* behavior. They may wander around the phase space for a long time but eventually their path will remain arbitrarily close to either a fixed point or a limit cycle. In this case the system is said to exhibit fixed point or limit cycle *attractors*. The system may wander ever further away from its starting point. Depending upon how the system behaves if the dynamics is run backwards in time, the system is said to exhibit fixed point or limit cycle *repellors*, or it may simply be divergent.

In spite of the astounding success of the reductionist program there is a growing realization that Nature is much more subtle than ever imagined. The whole may not merely be the sum of its parts, but may in fact exhibit behavior which transcends that of its components. This alternative viewpoint, termed *contextualism* by Cohen and Stewart, has begun to gain favor as a result of research being carried out in a wide variety of disciplines, including mathematics, physics, economics, biology, psychology, neuroscience and sociology, into the dynamics of what have come to be termed complex systems. These include cellular automata, spin glasses, economic systems, speciation, cognition, brains, and culture. Although many systems may be readily identified as complex, the concept of the complex system remains in its infancy, and there is as yet no clear consensus concerning its proper scope and definition. Indeed, one of the fundamental tasks of complex systems theory is to attempt to delineate those characteristics which clearly separate complex systems from merely complicated systems. Physics and mathematics have taken different, though subtly related, approaches to this problem. In each case the methods being used reflect those of the sub-discipline in which such complex systems were first identified. In the case of physics, this was condensed matter physics, and so the methods are predominantly the methods of statistical mechanics. These are described in detail in other articles in this encyclopedia and will not be discussed further here. This article will focus upon the approaches being taken within mathematics. There the first awareness of complexity appeared at the turn of the twentieth century with the discovery of *limitative* results and so it is there that we begin our journey.

The first major limitative result was obtained by the French mathematician, Henri Poincaré, in his studies of celestial mechanics. The motion of the heavens is governed to a remarkable degree of accuracy by Newton's laws of gravitation and motion. The solution of Newton's equations for the motion of two bodies, such as the Earth and the Sun, are simple ellipses, and their calculation is now a straightforward schoolbook exercise. However, the motion of three bodies, such as the Sun-Earth-Moon system, proved to be surprisingly difficult to analyze formally. Numerical methods provided predictions to high accuracy, but no closed form solutions, no single equations, could be written down which would describe this motion exactly. Poincaré demonstrated conclusively that, in general, no closed form solutions existed for the motion of more than two bodies. In large measure this failure was due to the presence of a hitherto unrecognized form of motion, chaos. In chaotic motion, if a system is started in two differing but nearby states, the subsequent paths which the system follows diverge -- that is, become ever more uncorrelated over time. As a result of this, long-term prediction of the behavior of the system becomes impossible.

The second major limitative result occurred in mathematical logic. Beginning with Hilbert, and pursued by Russell and Whitehead and many others, mathematicians and philosophers sought a set of axioms, a set of fundamental principles or laws, from which all mathematical truths could be derived using the principles of pure reason alone. Like their physicist counterparts, mathematicians believed in the reductionist approach, and in the existence of a fundamental theory of everything, which would be based upon a few simple laws or axioms. Suspicions that this might not be the case began with one of its staunchest supporters, the British philosopher, Bertrand Russell, who, while attempting to lay down a fundamental theory of sets, discovered his famous paradox. According to the tenets of mathematical logic, a true theory of mathematics should be consistent, that is, free from contradiction, and complete, meaning that every true mathematical statement should be found within this theory. This does not mean that every statement which can be formalized in mathematical terms should be true, but it does suggest that every mathematical statement should be either true or false. A paradox is an example of a statement which is neither true not false. It is indeterminate. A classic example of a paradox is the statement “The Cretan said, ‘All Cretans are liars’”. There is no way in which this statement can be assigned a value of true or false. A theory with a paradox is in peril, and Russell discovered just such a paradox in his theory of sets. In form it is quite similar to the previous example and goes like this. Let A be the set consisting of all sets which do not have themselves as members. That is, A is the set which consists of all of those sets B for which B is not a member of B. If A contains itself, then A is not a set which does not contain itself as a member, which means that A cannot contain itself, a contradiction. But if A does not contain itself as a member, then A meets its own condition for inclusion as a member of itself, again a contradiction. There is no way out.

The key to both paradoxes lies in their self-referential nature. Consequently one might think that one way to prevent this from happening is to ban self-reference. But nature abounds in self-reference and so do formal systems which have even the most basic degree of explanatory power. In the early 1930s, the mathematician Kurt Godel proved his famous incompleteness theorem, in which he showed that any formal logical system which was capable of expressing the basic laws of arithmetic (and it is hard to think of any useful system which could not do so) possessed a true statement which was not provable within the system. That is, the formal system had true statements but it was impossible to prove whether or not these statements were in fact true. Godel proved this using a self-referential construction of the kind described above. Self-reference and paradox were unavoidable.

At the same time that Godel was demonstrating the limits of logic, another mathematician, Alan Turing, was demonstrating the limits of computation. In essence, Turing developed a theory which attempted to formalize our intuitive idea of a computation, and what it means to compute something. His theory was framed in the most general of terms, and a central tenet of the modern theory of computation, the Church-Turing thesis, states that any alternative formulation of the notion of computation can ultimately be shown to be identical with that of Turing. Having provided a general structure theory of computation, Turing was then able to demonstrate that even some of the most fundamental questions of computation theory, such as whether or not a particular computational procedure will always yield an answer, are themselves unanswerable. Again, self-reference proved to be the key to the proof. There were limits to what could be computed. In turn, this meant that there were limits to what could be explicitly calculated and simulated. Thoughts that the computer might save us from the limitations of formal reasoning were dashed in the face of their own limitations.

We thus have three possible conceptions for the truly complex: systems whose behavior is unpredictable, systems whose laws are unprovable, and systems whose functions are noncomputable. However, there is another approach to complexity which suggests that these classes of systems are, shall we say, too complex to be considered as complex systems. This approach finds its roots in the work of Wolfram on *cellular automata.* Cellular automata provide what many consider to be the simplest formal models of complex systems. They consist of a collection of cells, which are arranged in a lattice, the dimension of the lattice being the dimension of the automaton.

The lattice may be infinite or finite in extent. At any given time, each cell may exist in one of a finite set of states. Each cell is assigned a transition rule which tells it how to choose its next state given its current state and the state of certain neighboring cells. These neighboring cells are themselves selected according to some fixed template. Usually the transition rule is the same for all cells (homogeneous cellular automaton) though sometimes it is allowed to vary over the lattice (inhomogeneous or disordered cellular automaton). The procedure for selecting the neighboring cells is generally fixed for all cells and the neighboring cells are usually those which are immediately adjacent (local cellular automaton) though sometimes they are chosen from those at a distance (non-local cellular automaton). In the case of one-dimensional cellular automata, the neighboring cells are those which lie within a fixed number of sites from the cell.

Wolfram engaged in an exhaustive search of all one-dimensional, two-state, three-neighbor, cellular automata. Based upon the behavior that he observed he proposed a classification of these automata into uniform, periodic, chaotic and complex. Uniform cellular automata evolve to a fixed point, while periodic automata exhibit periodic behavior. Chaotic cellular automata produce random appearing patterns, and in the case of an infinite lattice exhibit aperiodic trajectories. The complex cellular automata formed the remainder, and consist of those cellular automata which exhibit very long-lived transients. Wolfram conjectured that it was these latter cellular automata which were capable of carrying out meaningful computations. Moreover, if we choose one particular state, and define a new criterion termed the “lambda parameter”, which measures the percentage of times that the next state is this state, then we find that the Wolfram classification correlates roughly with the lamba parameter, so that the complex automata lie between the regular automata and the chaotic automata, hence the term edge of chaos. This suggests that complexity lies somewhere between the merely complicated and the truly impossible. Later work by Packard and Langton suggested that if a set of cellular automata was evolved using genetic algorithms, which mimic the natural process of evolution, where the survival or fitness of a particular cellular automaton is determined by its success in carrying out a predetermined, fixed computation, then the majority of the resulting automata will lie within the class of complex automata. It was suggested therefore that nature evolves computational systems so that they come to lie at the edge of chaos, and therefore become increasingly complex.

While this remains an intriguing and stimulating metaphor, subsequent research has cast doubt upon its generalizability. The patterns generated by such automata are exquisitely sensitive to noise or external inputs. Moreover there is the eye of the beholder problem as the classification depends to some degree upon an extrinsic judgment of the complexity, and therefore the value of the patterns generated by a cellular automaton. For these reasons we shall focus upon the more traditional approaches.

2. Structural and Functional Approaches

The structuralist approach to complex systems is similar to that of systematics in biology, an attempt to classify systems on the basis of similar and dissimilar structural features. From this perspective, the complexity of a system depends upon the number and diversity of its constituent parts, the number and diversity of the interconnections and the interactions which occur between and among these constituents, and the structure of the organization which weaves these constituents into a coherent whole. While this approach has had some success within the domain of biology, it has not proven as successful in other domains where the sheer diversity of structures is staggering. Moreover there does not appear to be any consistent relationship between the structure of a system and the behavior which it generates. Thus it is difficult at present to provide any useful structural paradigm for classifying and understanding complex systems. It is here where formal mathematical methods may prove to be of particular benefit. Formal models may be constructed free of the inevitable noise and variability inherent in natural systems and so provide a set of benchmarks or prototypes by means of which natural systems might be classified. Structural classification has proven to be a powerful tool in pure mathematics and may yet again prove itself here, but the subject is still in its infancy.

The functionalist approach attempts to classify complex systems by virtue of the particular forms of dynamical behaviors which they exhibit. At first glance this appears to be more promising than the structuralist approach, since there already appears to be a classification of behaviors into such groupings as fixed point, periodic, *quasiperiodic*, chaotic, turbulent. This classification applies only to deterministic *dynamical systems*. The behavior of stochastic dynamical systems is far more subtle, as has been discovered by those who have extended the concept of chaos into the realm of quantum mechanics. Moreover, these classifications apply to isolated systems which are not receiving any stimulation or energy from their environments. Virtually all complex systems of any practical importance exhibit strong couplings to their environments. Once the environment is allowed to play a role, a staggering diversity again appears.

A functional consideration of potentially great importance, though, is that of *emergence*. This is the idea that the behavior of a system at one level of scale gives rise to wholly unexpected or unpredicted behavior at a higher level of scale. The study of emergence is an active area of research currently within the field of complex systems theory, but many of the examples are of formal mathematical models of limited scope, and there is as yet no consensus on even the definition of emergence, let alone on its putative mechanisms, nor on how this might provide for a classification of complex systems.

Some notions, such as broken ergodicity (the exploration by a system of only a restricted portion of its phase space), broken symmetry (the failure of the behavior of a system to exhibit symmetries inherent in its defining equations) and related measures such as entropy, have begun to offer some promise, appealing not to specific local behaviors but rather to global characteristics. This is an active and promising area of research, but as of yet only weak correlations appear to exist between these functional considerations and our intuitive ideas of complexity.

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Biographical Sketch

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