STABILITY THEORY

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Summary

Stability plays an important role in the theory of dynamical systems and control. It characterizes the property of an unperturbed trajectory that all perturbed trajectories starting nearby stay nearby: small perturbations cause only small changes in the system behavior. The most important concept of stability has bee introduced by the Russian mathematician A. M. Lyapunov in 1892. Based on his famous work a general "Lyapunov theory" has been developed to investigate the stability behavior of general dynamical systems. Here, the basic ideas and results of this theory are presented.

1. Introduction

The property of stability is very important for the behavior of dynamical systems. Intuitively, stability can be understood as the requirement that small perturbations of the system cause only small changes of the system behavior. As perturbations external excitations or changes of the initial conditions can be considered as well. The development of stability concepts always tried to meet these intuitive ideas. But for a correct formulation of the stability problem exact definitions and criteria are required. While for linear time-invariant systems independent investigations were performed resulting in algebraic and geometric criteria, the stability analysis of nonlinear dynamical systems is still based on the famous work of the Russian mathematician A. M. Lyapunov (1857-1918) who presented a book on the general problem of the stability of motion in 1892. Lyapunov's stability theory shows the complexity of the stability problem requiring a very precise discussion of the problem.

The following stability analysis is based on a state space model of nonlinear dynamical systems

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), t) \tag{1}$$

or

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), t) + \mathbf{B}(\mathbf{x}(t), t)\mathbf{u}(t)\,, \tag{2}$$

$$\mathbf{y}(t) = \mathbf{c}(\mathbf{x}(t), t) + \mathbf{D}(\mathbf{x}(t), t)\mathbf{u}(t)\,. \tag{3}$$

For the free system (1) an uncontrolled nonlinear system or a closed-loop controlled system is considered. For system (2, 3) the input-output behavior is considered where $\mathbf{u}$ can be interpreted as external disturbances or as control inputs which have to be designed suitably; the output $\mathbf{y}$ represents usually the measurements.

For linear systems we can speak about the stability of the system in general, but for nonlinear systems we have to distinguish more precisely if the stability of an equilibrium point, a trajectory, a limit cycle or an arbitrary attractor is considered. If a particular solution $\mathbf{x}(t) = \mathbf{x}_{\mathrm{p}}(t)$ is assumed, if necessary with a suitable input function $\mathbf{u}(t) = \mathbf{u}_{\mathrm{p}}(t)$, satisfying (1) or (2, 3) respectively, then the equations can be reformulated with respect to the deviations from $\mathbf{x}_{\mathrm{p}}(t)$, $\mathbf{u}_{\mathrm{p}}(t)$:

$$\mathbf{x}(t) = \mathbf{x}_{\mathrm{p}}(t) + \bar{\mathbf{x}}(t)\,,\ \mathbf{u}(t) = \mathbf{u}_{\mathrm{p}}(t) + \bar{\mathbf{u}}(t)\,. \tag{4}$$

This leads to

$$\dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{a}}(\bar{\mathbf{x}}(t), t)\,,\ \bar{\mathbf{a}}(\mathbf{0}, t) = \mathbf{0} \tag{5}$$

with

$$\bar{\mathbf{a}}(\bar{\mathbf{x}}(t), t) = \mathbf{a}(\mathbf{x}_{\mathrm{p}}(t) + \bar{\mathbf{x}}(t), t) - \mathbf{a}(\mathbf{x}_{\mathrm{p}}(t), t) \tag{6}$$

or,

$$\bar{\bar{\mathbf{a}}}(\mathbf{0}, t) = \mathbf{0}\,, \tag{7}$$

$$\bar{\mathbf{y}}(t) = \bar{\mathbf{c}}(\bar{\mathbf{x}}(t), t) + \bar{\mathbf{D}}(\bar{\mathbf{x}}(t), t)\bar{\mathbf{u}}(t)\,,\ \bar{\mathbf{c}}(\mathbf{0}, t) = \mathbf{0} \tag{8}$$

with

$$\bar{\bar{\mathbf{a}}}(\bar{\mathbf{x}}(t), t) = \bar{\mathbf{a}}(\bar{\mathbf{x}}(t), t) + \left[\bar{\mathbf{B}}(\mathbf{x}_{\mathrm{p}} + \bar{\mathbf{x}}, t) - \mathbf{B}(\mathbf{x}_{\mathrm{p}}, t)\right]\mathbf{u}_{\mathrm{p}}(t)\,, \tag{9}$$

$$\bar{\mathbf{B}}(\bar{\mathbf{x}}(t), t) = \mathbf{B}(\mathbf{x}_{\mathrm{p}} + \bar{\mathbf{x}}, t)\,, \tag{10}$$

$$\bar{\mathbf{c}}(\bar{\mathbf{x}}(t), t) = \mathbf{c}(\mathbf{x}_{p} + \bar{\mathbf{x}}, t) - \mathbf{c}(\mathbf{x}_{\mathrm{p}}, t) + \left[\bar{\mathbf{D}}(\mathbf{x}_{\mathrm{p}} + \bar{\mathbf{x}}, t) - \mathbf{D}(\mathbf{x}_{\mathrm{p}}, t)\right]\mathbf{u}_{\mathrm{p}}(t)\,, \tag{11}$$

$$\bar{\mathbf{D}}(\bar{\mathbf{x}}(t),t) = \mathbf{D}(\mathbf{x}_{\mathrm{p}} + \bar{\mathbf{x}},t)\,, \tag{12}$$

$$\bar{\mathbf{y}}(t) = \mathbf{y}(t) - \mathbf{y}_{\mathrm{p}}(t)\,,\ \mathbf{y}_{\mathrm{p}}(t) = \mathbf{c}(\mathbf{x}_{\mathrm{p}},t) + \mathbf{D}(\mathbf{x}_{\mathrm{p}},t)\mathbf{u}_{\mathrm{p}}(t)\,. \tag{13}$$

By the transformation (4) the general stability investigation for the particular solution $\mathbf{x}_{\mathrm{p}}(t)$ is attributed to the stability analysis of the equilibrium point $\bar{\mathbf{x}} = \mathbf{0}$ of the system (5) of (7, 8). But it has to be accepted that a time-invariant system (1) or (2, 3) is transformed in a time-variant system (5) or (7, 8) for time-varying trajectories $\mathbf{x}_{\mathrm{p}}(t)$. Only special cases such as constant operation points $\mathbf{x}_{\mathrm{p}}(t) = \mathbf{x}_{\mathrm{p0}}$ lead to time-invariant systems (5) or (7, 8) if the original systems are time invariant, too.

In the following the descriptions (5) or (7, 8) are assumed. By that agreement the bars on the vectors in (5) and (7, 8) are dropped without being confused.

2. Linearization: Stability in the First Approximation

In many applications a feedback control is designed to stabilize an equilibrium point (or a particular motion). In case of additional disturbances only small deviations $\tilde{\mathbf{x}}(t)$ from the equilibrium point are allowed. As long as the nonlinearity functions are continuously differentiable in a neighborhood of the desired equilibrium point, then the system behavior may be approximated by the linearized equations

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}(t)\tilde{\mathbf{x}}(t) \tag{14}$$

or

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}(t)\tilde{\mathbf{x}}(t) + \mathbf{B}(t)\mathbf{u}(t)\,, \tag{15}$$

$$\tilde{\mathbf{y}}(t) = \mathbf{C}(t)\tilde{\mathbf{x}}(t) + \mathbf{D}(t)\mathbf{u}(t)\,. \tag{16}$$

The system matrices are designed as

$$\mathbf{A} := \left.\frac{\partial \mathbf{a}(\mathbf{x},t)}{\partial \mathbf{x}^{\mathrm{T}}}\right|_{\mathbf{x}=\mathbf{0}},\ \mathbf{B} := \mathbf{B}(\mathbf{0},t)\,,\ \mathbf{C} := \left.\frac{\partial \mathbf{c}(\mathbf{x},t)}{\partial \mathbf{x}^{\mathrm{T}}}\right|_{\mathbf{x}=\mathbf{0}},\ \mathbf{D} := \mathbf{D}(\mathbf{0},t) \tag{17}$$

where **A**, **C** are Jacobian matrices evaluated at $\mathbf{x} = \mathbf{0}$.

Very often Eqs. (15, 16) are the starting point of the stability analysis or the control design. This established procedure is justified by the method of first approximation.

Theorem 1: If the linearized system (14) is exponentially stable (cf. (20)) and

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\|\mathbf{a}(\mathbf{x},t) - \mathbf{A}(t)\mathbf{x}\|}{\|\mathbf{x}\|} = 0 \tag{18}$$

holds, then the equilibrium point $\mathbf{x} = \mathbf{0}$ of the nonlinear system (5) is exponentially stable as well.

Theorem 2: If the linearized system (15) is stabilized exponentially by a linear static or dynamic output feedback of (16) and

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\|\mathbf{c}(\mathbf{x},t)-\mathbf{C}(t)\mathbf{x}\|}{\|\mathbf{x}\|}=0 \tag{19}$$

holds additionally to (18), then the equilibrium point $\mathbf{x}=\mathbf{0}$ of the nonlinear system (7, 8) is exponentially stabilized by the same feedback.

The requirements (18, 19) mean that the deviations between the nonlinear and the linearized system are small of higher than first order. Additionally, we have to be aware of the effect that stability of the linearized system is a global property while for the nonlinear system the stability of the equilibrium point $\mathbf{x}=\mathbf{0}$ is a local property in its neighborhood. Exponential stability is only guaranteed in a domain of attraction which may be small. A control design according to Theorem 2 should also make this domain as large as possible.

The notion of exponential stability of $\mathbf{x}=\mathbf{0}$ means the requirement that any trajectory $\mathbf{x}(t)$ for arbitrary initial conditions $\mathbf{x}(0)=\mathbf{x}_0$ in a small but full neighborhood of $\mathbf{x}=\mathbf{0}$ satisfies the estimate

$$\|\mathbf{x}(t)\| \le \alpha \|\mathbf{x}_0\| \mathrm{e}^{-\beta t} \tag{20}$$

for certain positive constants α,β. For time-invariant system matrices $\mathbf{A}(t)=\mathbf{A}$ the system (14) is exponentially stable if and only if the conditions

$$\operatorname{Re}\lambda_i(\mathbf{A})<0,\ i=1,\ldots,n \tag{21}$$

hold where $\lambda_i(\mathbf{A})$ are the eigenvalues of $\mathbf{A}$. The test for (21) consists either in the calculation of the eigenvalues λ_i or in the application of one of the stability criteria for linear time-invariant systems.

For linear time-variant systems the proof of exponential stability is much more difficult. Sometimes there is a speculation with respect to the “frozen” eigenvalues $\lambda_i(t)$, $i=1,\ldots,n$,

$$\det\left[\lambda_i(t)\mathbf{I}-\mathbf{A}(t)\right]=0, \tag{22}$$

such that a condition $\operatorname{Re}\lambda_i(t)\le-\delta<0$ would guarantee exponential stability. But this speculation is definitely wrong which is shown by a simple counter-example.

Example 1: Consider the linear periodic system

$$\dot{x}_1 = \left(\frac{3}{2}\cos 3t - \frac{1}{2}\right)x_1 + \frac{3}{2}\left(1 - 3\sin t\right)x_2\,,\ \dot{x}_2 = -\frac{3}{2}\left(1 + \sin 3t\right)x_1 - \left(\frac{3}{2}\cos 3t + \frac{1}{2}\right)x_2 \qquad (23)$$

The frozen eigenvalues are constant: $\lambda_i = \lambda_2 = -\dfrac{1}{2}$. Nevertheless the system (23) is exponentially unstable because of the solution

$$x_1(t) = e^t \cos\frac{3}{2}t \cdot x_{10} + e^{-2t}\sin\frac{3}{2}t \cdot x_{20}\,,\ x_2(t) = -e^t \sin\frac{3}{2}t \cdot x_{10} + e^{-2t}\cos\frac{3}{2}t \cdot x_{20}\,. \qquad (24)$$

The stability analysis of time-variant systems is difficult in general. In the special case of linear periodic systems Floquet's theory can be applied, which offers a systematic numerical approach for the stability check at least.

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Biographical Sketch

Peter C. Müller, born in 1940 at Stuttgart, Germany. He got degree of "Diplom-Mathematiker" in 1965 from the Technical University of **Stuttgart**

Profession:

1966 - 1981	Institute of Mechanics, Technical University of **München**
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1978	Professor of Mechanics
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Membership:
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1988 – 2000: Selection Committee of the Alexander von Humboldt Foundation
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