FREQUENCY DOMAIN SUBSPACE ALGORITHMS

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Contents

Summary

In this chapter we discuss subspace algorithms using the covariance matrix of the disturbing noise. The performance of these algorithms is compared with the (weighted) linear least squares and (weighted) generalized total least squares methods discussed in *Estimation with known Noise Model*. In turns out that the frequency domain subspace identification algorithms are very good alternatives to generate high quality starting values for the optimal maximum likelihood solution.

1. Introduction

In McKelvey *et al.* (1996) and Van Overschee and De Moor (1996) frequency domain subspace algorithms have been developed for respectively discrete-time and continuous-time models. These identification methods have proven to be very effective in solving real life problems such as, for example, modal analysis (McKelvey *et al.*, 1996), modeling of power transformers (Akcay *et al.*, 1999), flight flutter analysis and modeling of synchronous machines (Pintelon and Schoukens, 2001a).

In general (non-uniformly spaced frequency domain data and/or arbitrarily colored disturbing noise) these algorithms are consistent only if the covariance matrix of the disturbing noise is known. Therefore, instrumental variable based versions have been developed which are consistent without requiring the knowledge of the noise covariance matrix (see McKelvey, 1997 for discrete-time models and Yang and Sanada, 2000 for continuous-time models). Since the algorithms using the noise covariance matrix have better statistical properties than the instrumental variable based versions, and since the required noise information can easily be obtained from a small number of independent repeated experiments (see *Estimation with unknown Noise Models*), we limit the discussion to the methods requiring the noise covariance matrix.

The chapter is organized as follows. Section 2 develops the basic model equations (plant and noise models) used by the frequency domain subspace algorithms. A detailed description of the algorithms using the true noise covariance matrix is given in Section 3. The assumptions commonly made in subspace identification are that the input is exactly known and that the system is proper.

What to do if these assumptions are not met is discussed in Section 4. Section 5 compares the performance of the subspace algorithms with some of the estimators discussed in *Estimation with known Noise Model*. Finally, Section 6 illustrates the approach on a real measurement example. To simplify the notations we limit the discussion to single input, single output systems. Extension of the results to multivariable systems is straightforward (McKelvey *et al.*, 1996 and Van Overschee and De Moor, 1996).

2. Model Equations

2.1. Plant Model

Consider a proper, n_ath order single input single output system. The relation between the input $u(t)$ and the output $y(t)$ can be written under state space representation form as, respectively,

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

for continuous-time systems, and

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{2}$$

for discrete-time systems, where $x(t) \in \mathrm{R}^{n_a}$ is the state vector. The frequency domain subspace algorithms estimate the parameters $A \in \mathrm{R}^{n_a \times n_a}$, $B \in \mathrm{R}^{n_a \times 1}$, $C \in \mathrm{R}^{1 \times n_a}$ and $D \in \mathrm{R}$ from a transformed version of the state space equations (1) and (2). These are constructed as follows.

Assume that the input is periodic and that the steady state response over an integer number of periods is observed. The discrete Fourier transform (DFT) of (1) and (2) then becomes

$$\begin{aligned} \xi_k X(k) &= AX(k) + BU(k) \\ Y(k) &= CX(k) + DU(k) \end{aligned} \tag{3}$$

with $Z(k)$, $Z = U, Y, X$, the DFT of $z(t)$, $z = u, y, x$

$$Z(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} z(t) e^{-j2\pi kt/N} \tag{4}$$

and where $\xi = z$ for discrete-time systems and $\xi = s$ for continuous-time systems. Recursive use of the second and the first equation of (3) gives

$$\begin{aligned} \xi_k^p Y(k) &= \xi_k^{p-1}(C\xi_k X(k) + D\xi_k U(k)) \\ &= \xi_k^{p-1}(CAX(k) + CBU(k) + D\xi_k U(k)) \\ &= \ldots \\ &= CA^p X(k) + (CA^{p-1}B + CA^{p-2}B\xi_k + \ldots + CB\xi_k^{p-1} + D\xi_k^p)U(k) \end{aligned} \tag{5}$$

Writing the last equation of (5) for $p = 0, 1, \ldots, r-1$ $(r > n_a)$ on top of each other gives

$$W_r(k)Y(k) = O_r X(k) + S_r W_r(k)U(k) \tag{6}$$

with

$$W_r(k) = \begin{bmatrix} 1 \\ \xi_k \\ \ldots \\ \xi_k^{r-1} \end{bmatrix}, O_r = \begin{bmatrix} C \\ CA \\ \ldots \\ CA^{r-1} \end{bmatrix} \text{and}\, S_r = \begin{bmatrix} D & 0 & \ldots & 0 & 0 \\ CB & D & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ CA^{r-2}B & CA^{r-3}B & \ldots & CB & D \end{bmatrix} \tag{7}$$

Collecting (6) for $k = 1, 2, \ldots, F$ gives

$$\mathbf{Y} = O_r \mathbf{X} + S_r \mathbf{U} \tag{8}$$

with

$$\begin{aligned} \mathbf{Y} &= \left[W_r(1)Y(1)\; W_r(2)Y(2) \ldots W_r(F)Y(F) \right], \\ \mathbf{U} &= \left[W_r(1)U(1)\; W_r(2)U(2) \ldots W_r(F)U(F) \right], \\ \mathbf{X} &= \left[X(1)\; X(2) \ldots (F) \right]. \end{aligned} \tag{9}$$

The complex data matrices $\mathbf{Y}$ and $\mathbf{U}$ have r rows and F columns. $\mathbf{X}$ is a complex n_a by F matrix, and O_r and S_r are, respectively, real r by n_a and r by r matrices. Equation (8) is converted in a real set of equations as

$$\mathbf{Y}^{\mathrm{re}} = O_r \mathbf{X}^{\mathrm{re}} + S_r \mathbf{U}^{\mathrm{re}}, \tag{10}$$

where $(\ )^{\mathrm{re}}$ locates the real and imaginary parts beside each other, for example,

$$\mathbf{Y}^{\mathrm{re}} = [\mathrm{Re}(\mathbf{Y})\ \ \mathrm{Im}(\mathbf{Y})] \tag{11}$$

Equation (10) with r larger than the model order n_a, is the basic model used in frequency domain subspace identification.

The extended observability matrix O_r has the following shift property

$$O_{r[1:r-1,:]} A = O_{r[2:r,:]}, \tag{12}$$

which will be used in the identification procedure. O_r is not unique since it depends on the choice of the state variables. Indeed, replacing (A, B, C, D, X) by $(T^{-1}AT, T^{-1}B, CT, D, T^{-1}X)$, with T an invertible matrix, in the state space equations (3), does not change the input-output transfer function

$$G(\xi) = C(\xi I_{n_a} - A)^{-1} B + D \tag{13}$$

but does change O_r to $O_r T$. Note that $O_r \mathbf{X}$ and S_r in model equation (8) are invariant w.r.t. the invertible transformation T.

Since A, B and C are not unique, one may wonder how the quality of the estimates $\hat{A}$, $\hat{B}$ and $\hat{C}$ can be evaluated. This is possible by referring the estimates $\hat{A}$, $\hat{B}$ and $\hat{C}$ to one particular (true or noisy) state space realization A, C

$$\hat{A}_T = T^{-1}\hat{A}T,\ \hat{B}_T = T^{-1}\hat{B} \text{ and } \hat{C}_T = \hat{C}T \text{ with } T = \hat{O}_r^{+} O_r, \tag{14}$$

where $+$ is the Moore-Penrose pseudo-inverse $(\hat{O}_r^{+} = (\hat{O}_r^T \hat{O}_r)^{-1} \hat{O}_r^T)$, and where O_r, $\hat{O}_r$ are defined as in (7) using, respectively, (A, C) and $(\hat{A}, \hat{C})$. Note that applying a similarity transformation P to $\hat{A}$, $\hat{B}$ and $\hat{C}$ does not change $\hat{A}_T$, $\hat{B}_T$ and $\hat{C}_T$. Hence, it is possible to calculate the sample mean and sample covariance matrices of $\hat{A}_T$, $\hat{B}_T$ and $\hat{C}_T$. To simplify the notations the subscript T will be dropped in the sequel of the paper.

For identifiability purposes it will be assumed that the state space realization (3) is observable, $\mathrm{rank}(O_{\mathrm{r}}) = n_a$ for any $r \geq n_a$, and controllable, $\mathrm{rank}([B\ \ AB\ \ \ldots\ \ A^{q-1}B]) = n_a$ for any $q \geq n_a$.

3 . Noise Model

The theory is developed assuming that the input is exactly known and that the output is observed with errors

$$\begin{aligned} U(k) &= U_0(k) \\ Y(k) &= Y_0(k) + N_Y(k) \end{aligned} \qquad (15)$$

with $U_0(k)$, $Y_0(k)$ the true input and output DFT spectra, and $N_Y(k)$ the noise errors. $N_Y(k)$ has zero mean $E\{N_Y(k)\} = 0$, variance $\sigma_Y^2(k) = \text{var}(N_Y(k)) = E\{|N_Y(k)|^2\}$, and is independent of $Y_0(k)$. What to do if also the input observations are noisy is discussed in Section 4, Practical Remarks. For noisy output DFT spectra $Y(k)$, model (10) becomes

$$\mathbf{Y}^{\text{re}} = O_r\mathbf{X}^{\text{re}} + S_r\mathbf{U}^{\text{re}} + \mathbf{N}_{\mathbf{Y}}^{\text{re}}, \qquad (16)$$

where $\mathbf{N_Y}$ has the same structure as $\mathbf{Y}$ in (9).

3. Subspace Algorithms

Subspace identification algorithms are basically a three step procedure. First, an estimate $\hat{O}_r$ of the extended observability matrix is obtained using model (16). This is the most difficult step and consists mainly of eliminating the term depending on the input and reducing the noise influence.

Next, $\hat{A}$ and $\hat{C}$ are found as the least squares solution of the overdetermined set of equations (12) and as the first row of $\hat{O}_r$ (see (7)) respectively. Finally, $\hat{B}$ and $\hat{D}$ are found as the linear least squares solution of

$$V_{\text{SUB}}(C, D, \hat{A}, \hat{C}, Z) = \sum_{k=1}^{F} W^2(k) \left| Y(k) - [\hat{C}(\xi_k I_{n_a} - \hat{A})^{-1} B + D] U(k) \right|^2, \qquad (17)$$

where $W(k)$ is a well chosen real weighting function.

We present two algorithms, one for discrete-time systems $(\xi = z)$, based on McKelvey *et al.*, (1996), and one for continuous-time system $(\xi = s)$, based on Van Overschee and De Moor (1996). The numerically efficient implementation of these algorithms is due to Verhaegen (1994).

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Biographical Sketches

Rik Pintelon was born in Ghent, Belgium, on December 4, 1959. He received the degree of electrical engineer (burgerlijk ingenieur) in July 1982, the degree of doctor in applied sciences in January 1988, and the qualification to teach at university level (geaggregeerde voor het hoger onderwijs) in April 1994, all from the Vrije Universiteit Brussel (VUB), Brussels, Belgium. From October 1982 till September 2000 he was a researcher of the Fund for Scientific Research – Flanders at the VUB. Since October 2000 he is professor at the VUB in the Electrical Measurement Department (ELEC). His main research interests are in the field of parameter estimation / system identification, and signal processing.

Johan Schoukens was born in Belgium in 1957. He received the degree of engineer in 1980 and the degree of doctor in applied sciences in 1985, both from the Vrije Universiteit Brussel. The prime factors of his interest are in the filed of system identification for linear and nonlinear systems and growing tomatoes in his green house.