VOLTERRA AND FLIESS SERIES EXPANSION

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Summary

After recalling the definition of the Volterra series expansion and some of its convergence issues, we will study various methods in order to derive the Volterra kernels and the response to typical inputs. In particular, an algorithm, by means of which a large class of nonlinear systems can be analysed, is described.

This algorithm is based on an algebraic approach to Volterra functional expansions using noncommutative generating power series; this approach allows a natural generalization, to the nonlinear domain, of the symbolic operational calculus of Heaviside, widely used in linear system theory. Moreover, it has the advantage, compared with the method using the multidimensional Laplace transforms, of allowing an easy implementation on a computer.

1. Introduction

The input-output behavior of a large class of nonlinear systems can be described by the Volterra functional expansion

$$y(t) = \int_0^t h_1(t,\tau_1)u(\tau_1)d\tau_1 + \int_0^t\int_0^{\tau_2} h_2(t,\tau_1,\tau_2)u(\tau_1)u(\tau_2)d\tau_1 d\tau_2 + \ldots,$$

where $y(t)$ is the circuit output and $u(t)$ is the circuit input (assumed to be here scalar for simplicity sake). h_n is the *n*th-order Volterra kernel. This expansion is a generalisation of the well-known convolution integral

$$y(t) = \int_0^t h_1(t, \tau_1) u(\tau_1) d\tau_1$$

used in linear system theory.

These expansions are used in every branch of nonlinear system theory: identification and modeling, realization, stability, optimal control, stochastic differential equations, and filtering.

Although the Volterra series has been successfully used in many applications, it has not received a great deal of attention from engineers and designers. The reason for this seems to be the tedious computations involved in the determination of Volterra kernels. The algebraic approach to nonlinear functional expansions based on noncommutative generating power series, offers a powerful and systematic tool for analyzing a large class of nonlinear systems.

After recalling the definition of the Volterra series expansion and some of its convergence issues, we will study various methods in order to derive the Volterra kernels and the response to typical inputs. The analysis is then applied to the study of weakly nonlinear circuits in order to derive distortion rates or intermodulation products.

2. Functional Representation of Nonlinear Systems

2.1. Volterra Functional Series

For the simplicity of presentation, we shall consider time-invariant systems. If a system is linear and time-invariant, then the output $y(t)$ can be expressed as the convolution of the input $u(t)$ with the system unit impulse response $h(t)$:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau \tag{1}$$

The system unit impulse response $h(t)$ completely characterizes the linear time-invariant system since, once known, the response to any input can be determined from (1). A system is said to be causal if the output at any given time does not depend on future values of the input. That is, for any time t_1,

$$y(t_1) = \int_{-\infty}^{0} h(\tau) u(t_1 - \tau) d\tau = 0 .$$

This will be so if and only if

$$h(\tau) = 0, \qquad \text{for} \qquad \tau < 0 .$$

The extension of (1) to nonlinear time-invariant systems with memory is the Volterra series

$$y(t) = h_0 + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, ..., \tau_n) u(t-\tau_1) u(t-\tau_2) \cdots u(t-\tau_n) d\tau_1 d\tau_2 \cdots d\tau_n \qquad (2)$$

This functional form was first studied by Volterra. Much of his work in this area is summarized in his book. The functions $h_n(\tau_1, \tau_2 ..., \tau_n)$ are called the Volterra kernels of the system. A nonlinear system which can be represented by a Volterra series is completely characterized by its Volterra kernels.

Also, with an argument similar to that of linear systems, it can be shown that the nonlinear system is causal if and only if

$$h_n(\tau_1, \tau_2, ..., \tau_n) = 0, \quad \text{for } \tau_j < 0, \quad j = 1, ..., n .$$

It is well known that, without loss of generality, the kernels can be assumed to be symmetric. In fact any kernel $h_n(\tau_1, \tau_2, ..., \tau_n)$ can be replaced by a symmetric one by setting

$$h_n^{sym}(\tau_1, \tau_2, ..., \tau_n) = \frac{1}{n!} \sum_{(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_n}) \in S} h_n(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_n}),$$

where S is the set of all permutations of $\tau_1, ..., \tau_n$.

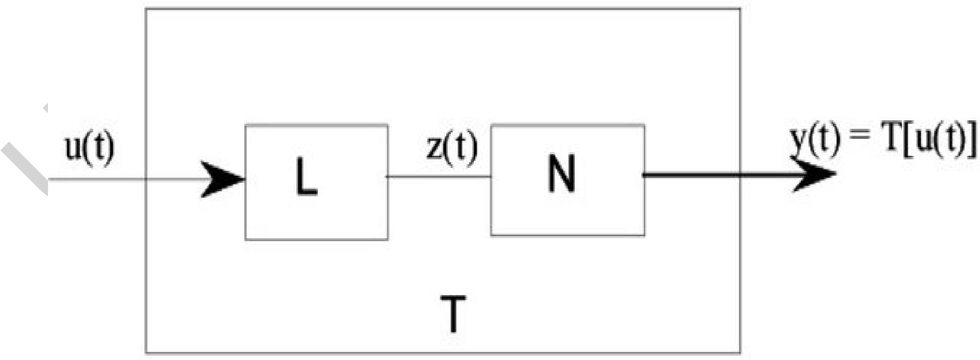

Figure 1: An example of a nonlinear system

The multiple Laplace transform $L[.]$ of the *n*th-order Volterra kernel $n > 0$ (one-sided in each variable)

$$H_n(s_1, ..., s_n) = \int_0^{\infty} \cdots \int_0^{\infty} h_n(\tau_1, ..., \tau_n) e^{-s_1\tau_1} \cdots e^{-s_n\tau_n} d\tau_1 d\tau_2 \cdots d\tau_n$$

is called the *n*th-order transfer function. Since $h_n(\tau_1,...,\tau_n)$ is symmetric, so is $H_n(s_1,...,s_n)$.

2.2. On the Convergence of Volterra Series

The Volterra series is a power series with memory. This can be seen by changing the input by a gain factor c so that the new input is $cu(t)$. By using (2), the new output is

$$y(t) = h_0 + \sum_{n=1}^{\infty} c^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\tau_2,...,\tau_n) u(t-\tau_1)u(t-\tau_2)\cdots u(t-\tau_n) d\tau_1 d\tau_2 \cdots d\tau_n,$$

which is a power series in the amplitude factor c. It is a series with memory since the integrals are convolutions. As a consequence of its power series character, there are some limitations associated with the application of the Volterra series to nonlinear problems. One major limitation is the convergence of this series.

In order to illustrate this let us consider the system of Figure 1 where the system L is a linear system with the unit impulse response $h(t)$

$$z(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau \qquad (3)$$

and the system N is a nonlinear no-memory system with the input-output relation

$$y(t) = N[z(t)] = \frac{z(t)}{1+z^2(t)}.$$

The Taylor series expansion of this expression is

$$y(t) = \sum_{n=0}^{\infty} (-1)^n [z(t)]^{2n+1} \qquad (4)$$

which converges only for $z^2(t) < 1$. The Volterra series representation of the overall system T is now easily derived by substituting (3) for (4) to obtain

$$y(t) = \sum_{n=0}^{\infty} (-1)^n \left[\int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau \right]^{2n+1},$$

in which the Volterra kernels are

$$h_{2n+1}(\tau_1,...,\tau_{2n+1}) = (-1)^n h(\tau_1)h(\tau_2)...h(\tau_{2n+1})$$

and

$$h_{2n}(\tau_1,...,\tau_{2n}) = 0, \quad n \geq 0.$$

Since the Taylor series converges only for $z^2(t) < 1$, the above Volterra series will diverge at those times for which $|z(t)| \geq 1$. The Volterra series thus is valid only for the class of inputs $u(t)$ for which the amplitude of $z(t)$ is less than one.

Now let N be replaced by the following nonlinear no-memory system

$$y(t) = E\text{sign}[z(t)].$$

Clearly, the system T cannot be represented by a Volterra series. It is therefore evident that generally many types of nonlinear systems, such as those that include saturating elements, cannot be characterized by a Volterra series that converges for all inputs.

Under certain conditions, a functional $y(t) = T[x(t)]$ can be approximated to any desired degree of accuracy by a finite series of the form of equation (2). Such a functional is called continuous. In particular it is easy to show that the functional relation between the solution (output) and the forcing function (input) of a nonlinear differential equation with constant coefficients which satisfies the Lipschitz conditions is continuous.

If $T[x(t)]$ can exactly be represented by a converging infinite series of the form of equation (2), it is called *analytic* or *weak*. Conditions for convergence are discussed by Volterra and Brillant. Brillant also notes that two special types of systems, for which the functional relation between input and output is analytic, are a linear system and a nonlinear no-memory systems with a power series relation between input and output. He then shows that various combinations such as cascading, adding, or multiplying such systems results in an analytic system.

In practice, most of the analogue circuits used in communication systems, such as modulators, mixers, amplifiers, harmonic oscillators, etc…, are of a weak nature and therefore analysed and designed in the frequency domain. For such weakly nonlinear circuits (having, say, distortion components of 20dB or more below the fundamental one), the Volterra series technique can be readily used in the frequency domain to obtain results both quantitatively and qualitatively.

Given an input-output map described by a nonlinear control system $\dot{x} = f(x,u)$ and a nonlinear output $y = h(x)$, there exists simple means for obtaining a series representation of the output $y(t)$ in terms of the input $u(t)$. When the control enters linearly, $\dot{x} = f(x) + ug(x)$, the method yields the existence of a Volterra series representation.

3. Recursive Computation of the Kernels

Several methods have been developed in the literature for determining the kernels or the associated transfer functions.

Among them, the method of exponential inputs is particularly used. After recalling this method we describe a differential geometry approach and an algebraic approach based on generating power series when the system is described by a set of differential equations. We shall see that the algebraic approach has the advantage of being easily implementable on a computer by using algebraic computing software.

3.1. Exponential Input Method

Let us consider the Volterra series expansion of a nonlinear system of the form

$$y(t)=\sum_{n=1}^{\infty}\int_0^{t_1}\cdots\int_0^{t_n} h_n(\tau_1,\tau_2,...,\tau_n)u(t-\tau_1)u(t-\tau_2)\cdots u(t-\tau_n)d\tau_1 d\tau_2\cdots d\tau_n\,. \qquad (5)$$

Let the input $u(t)$ be a sum of exponentials

$$u(t)=e^{s_1 t}+e^{s_2 t}+\cdots+e^{s_k t},$$

where $s_1,s_2,...,s_k$ are rationally independent. This means that there are no rational numbers $\alpha_1,\alpha_2,...,\alpha_k$ such that the sum $\alpha_1 s_1+\alpha_2 s_2+\cdots\alpha_k s_k$ is rational. Then (5) becomes

$$y(t)=\sum_{n=1}^{\infty}\left[\sum_{k_1=1}^{k}\cdots\sum_{k_n=1}^{k} H_n(s_{k_1},...,s_{k_n})e^{(s_{k_1}+\cdots+s_{k_n})t}\right]. \qquad (6)$$

If each s_i occurs in $(s_{k_1},...,s_{k_n}), m_i$ times, then there are

$$\frac{n!}{m_1!m_2!\cdots m_k!}$$

identical terms in the expression between brackets. Thus (5) can be written in the form

$$y(t)=\sum_{n=1}^{\infty}\sum_{m}\frac{n!}{m_1!m_2!\cdots m_k!}H_n(s_{k_1},...,s_{k_n})e^{(s_{k_1}+\cdots+s_{k_n})t}, \qquad (7)$$

where m under the summation sign indicates that the sum includes all the distinct vectors $(m_1,...,m_k)$ such that $\sum_{i=1}^{k} m_i=n$. Note that if $m_1=m_2=\cdots=m_k=1$ then the amplitude associated with the exponential component $e^{(s_{k_1}+\cdots+s_{k_n})t}$ is simply $k!H_k(s_1,...,s_k)$.

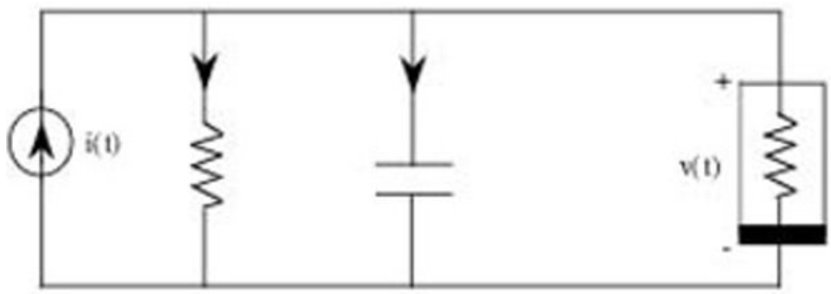

Figure 2: A simple nonlinear circuit

This suggests a recursive procedure for determining all the nonlinear transfer functions from the behavior of a system.

Let us apply the method to the simple nonlinear circuit of Figure 2 consisting of a capacitor, a linear resistor and a nonlinear resistor in parallel with the current source $i(t)$.

The nonlinear differential equation relating the current excitation $i(t)$ and the voltage $v(t)$ across the capacitor is given by

$$\dot{v} + k_1 v + k_2 v^2 = i \qquad (8)$$

Let $i(t) = e^{st}$. Equating the coefficients of e^{st} on both sides of (8) after the substitution of (7) for $v(t)$ we get

$$H_1(s) = \frac{1}{s + k_1}.$$

In order to determine $H_2(s_1, s_2)$ let us take $i(t) = e^{s_1 t} + e^{s_2 t}$ and identify the coefficient of the term $2!e^{(s_1+s_2)t}$ after the substitution of (7)for $v(t)$ in both sides of (8). We obtain $H_2(s_1, s_2)$ in term of $H_1(s)$ as follows

$$H_2(s_1, s_2) = -k_2 H_1(s_1) H_1(s_2) H_1(s_1 + s_2).$$

Similarly the third-order transfer function is obtained by injecting a sum of three exponentials inputs

$$i(t) = e^{s_1 t} + e^{s_2 t} + e^{s_3 t}.$$

It follows

$$H_3(s_1, s_2, s_3) = -\frac{2}{3}[H_2(s_1, s_2)H_1(s_3) + H_2(s_2, s_3)H_1(s_1) + H_2(s_1, s_3)H_1(s_2)]H_1(s_1 + s_2 + s_3)$$

Repeating this process indefinitely gives higher order nonlinear transfer functions in terms of lower-order nonlinear transfer functions.

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Biographical Sketch

Francoise Lamnabhi-Lagarrigue is Directeur de Recherche au Centre National de la Recherche Scientifique since 1993. She obtained the "maitrise des sciences mathematiques pures" degree at the Universite Paul Sabatier (Toulouse) in 1976 and she held a CNRS position in 1980. She obtained her Docteur d'Etat es Sciences Physiques from Universite Paris Sud in 1985. Some recent activities include the following :

- Editor of International Journal of Control;
- Responsible (1998-2001) of an agreement with the Groupe PSA on the Global Chassis Control project;
- Coordinator (1998-2002) of the Nonlinear Control Network (NCN)

  http://www.supelec.fr/lss/NCN;
- Director of the Marie Curie Control Training Site (CTS) http://www.supelec.fr/lss/CTS;
- Expert evaluator at the European Commission;
- Nominated member of the Board of Governors of the IEEE Control Systems Society for the year 2002 and elected Member for 2003-2005.

Her main research interests lie in the fields of nonlinear systems including system analysis and control design. Several important contributions should be emphasized. The analysis of nonlinear systems using functional expansions based on the Fliess generating power series and the development of formal computing for the Volterra series expansion associated with the output of nonlinear systems. Then singular optimal control and singular tracking for nonlinear systems have been studied. In the same time, an exact combinatorial formula for the derivatives of the output in terms of the derivatives of the inputs has been derived. More recently, her research interests include performance and robustness issues in nonlinear control, identification of nonlinear systems, and the application of these techniques to power systems, to vehicle global chassis control, to magnetic suspension, and to hydraulic actuators.

Francoise Lamnabhi-Lagarrigue is author and co-author of more than 100 publications which include one book, 50 journal papers and chapters of book, a Special issue of the International Journal on Control (Recent advances in the control of nonlinear systems, Vol 71(5), 1998), the co-edition of 6 books (the more recent ones being: - Stability and Stabilization of Nonlinear Systems, LNCIS 246, Springer-Verlag, 1999; - Nonlinear Control in the Year 2000, LNCIS 258 et LNCIS 259, Springer-Verlag, 2000; - Advances in Control of Nonlinear Systems, LNCIS 264, Springer-Verlag, 2001; - Commande des Systemes Non Lineaires, 2 volumes, Traite IC2, Hermes, 2002), about 45 published conference papers, 2 theses and various reports and European proposals.