THE PLANETARY *N*-BODY PROBLEM

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Contents

Summary

Planetary systems, under suitable general assumptions, admit positive measure sets of "initial data" whose evolution gives rise to the planets revolving on nearly circular and nearly co–planar orbits around their star. This statement (or more primitive formulations) challenged astronomers, physicists and mathematicians for centuries. In this chapter we shall review the mathematical theory (with particular attention to recent developments) needed to prove the above statement.

1. The *N*–Body Problem: A Continuing Mathematical Challenge

The problem of the motion of $N \geq 2$ point–masses (i.e., ideal bodies with no physical dimension identified with points in the Euclidean three–dimensional space) interacting only through Newton's law of mutual gravitational attraction, has been a central issue in astronomy, physics and mathematics since the early developments of modern calculus. When $N = 2$ the problem has been completely solved ("integrated") by Newton: the motion take place on conics, whose focus is occupied by the center of mass of the two bodies; but for $N \geq 3$ a complete understanding of the problem is still far away.

While the original impulse, coming from astronomy, has been somehow shaded by the massive use of machines for computing orbits of celestial bodies or satellites, the mathematical richness and beauty of the N –body problem has retained most of its original attraction; for a selection of recent contributions, see, e.g., (Chenciner and

Montgomery, 2000), (Ferrario and Terracini, 2004), (Hampton and Moeckel, 2006), (Chen, 2007), (Fusco, Gronchi and Negrini, 2011), (Chierchia and Pinzari, 2011 (c)).

Here, we will be concerned with the *planetary* N*–body problem*, which, as the name says, deals with the case of one body (the "Sun" or the "Star") having mass much bigger than the remaining bodies ("planets"). The main question is then to determine "general" conditions under which the planets revolve around the Sun without collisions and in a "regular way" so that, in particular, no planet crashes onto another planet or onto the Sun, nor does it escape away from such "solar system".

Despite the efforts of Newton, Euler, d'Alembert, Lagrange, Laplace and, especially, Henri Poincaré and G.D. Birkhoff, such question remained essentially unanswered for centuries. It is only with the astonishing work of a 26–year–old mathematician, V.I. Arnold (1937–2010), that a real breakthrough was achieved. Arnold, continuing and extending fundamental analytical discoveries of his advisor A.N. Kolmogorov on the so called "small divisors" (singularities appearing in the perturbative expansions of orbital trajectories), stated in 1963 (Arnold, 1963) a result, which may be roughly formulated as follows (*verbatim* formulations are given in Section 3.1 below).

If the masses of the planets are small enough compared to the mass of the Sun, there exists, in the phase space of the planetary N*–body problem, a bounded set of positive Lebesgue measure corresponding to planetary motions with bounded relative distances; such motions are well approximated by Keplerian ellipses with small eccentricities and small relative inclinations.*

Arnold gave a brilliant proof in a special case, namely, the planar three–body problem (two planets), giving some suggestions on how to generalize his proof to the general case (arbitrary number of planets in space). However, a complete generalization of his proof turned out to be quite a difficult task, which took nearly another fifty years to be completed: the first complete proof, based on work by M.R. Herman, appeared in Féjoz (2004) and a full generalization of Arnold's approach in Chierchia and Pinzari (2011c).

The main reason beyond the difficulties which arise in the general spatial case, is related to the presence of certain "secular degeneracies" which do not allow a *tout court* application of Arnold's "fundamental theorem" (see Section 3.2) to the general planetary case.

In this chapter we shall give a brief account (avoiding computations) of these results trying to explain the main ideas and technical tools needed to overcome the difficulties involved.

2. The Classical Hamiltonian Structure

2.1. Newton Equations and Their Hamiltonian Version

The starting point is with the Newton's equations for $1+n$ bodies (point masses), interacting only through gravitational attraction:

$$\ddot{u}^{(i)} = \sum_{\substack{0\le j\le n \\ j\ne i}} \mathrm{m}_j \frac{u^{(j)} - u^{(i)}}{|u^{(i)} - u^{(j)}|^3}, \qquad i = 0,1,\dots,n, \tag{1}$$

where $u^{(i)} = (u_1^{(i)}, u_2^{(i)}, u_3^{(i)}) \in \mathbb{R}^3$ are the Cartesian coordinates of the i^{th} body of (unscaled) mass $\mathrm{m}_i > 0$, $|u| = \sqrt{u\cdot u} = \sqrt{\sum_i u_i^2}$ is the standard Euclidean norm, "dots" over functions denote time derivatives, and the gravitational constant has been set to one (which is possible by rescaling time t).

Equations (1) are invariant by change of "inertial frames", i.e., by change of variables of the form $u^{(i)} \to u^{(i)} - (a+ct)$ with fixed $a, c \in \mathbb{R}^3$. This allows us to restrict the attention to the manifold of "initial data" given by

$$\sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(0) = 0, \qquad \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}(0) = 0; \tag{2}$$

indeed, just replace the coordinates $u^{(i)}$ by $u^{(i)} - (a+ct)$ with

$$a := \mathrm{m}_{\text{tot}}^{-1} \sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(0) \qquad \text{and} \qquad c := \mathrm{m}_{\text{tot}}^{-1} \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}(0), \qquad \mathrm{m}_{\text{tot}} := \sum_{i=0}^{n} \mathrm{m}_i .$$

The total linear momentum $\mathrm{M}_{\text{tot}} := \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}$ does not change along the flow of (1), i.e., $\dot{\mathrm{M}}_{\text{tot}} = 0$ along trajectories; therefore, by (2), $\mathrm{M}_{\text{tot}}(t)$ vanishes for all times. But, then, also the position of the barycenter $\mathrm{B}(t) := \sum_{i=0}^{n} m_i \mathrm{u}^{(i)}(t)$ is constant ($\dot{\mathrm{B}} = 0$) and, again by (2), $\mathrm{B}(t) \equiv 0$. In other words, the manifold of initial data (2) is invariant under the flow (1).

Equations (1) may be seen as the Hamiltonian equations associated to the Hamiltonian function

$$\mathcal{H}_{\mathrm{N}} := \sum_{i=0}^{n} \frac{|U^{(i)}|^2}{2\mathrm{m}_i} - \sum_{0\le i<j\le n} \frac{\mathrm{m}_i \mathrm{m}_j}{|u^{(i)} - u^{(j)}|},$$

where the subscript N signifies Newton, $(U^{(i)}, u^{(i)})$ are standard symplectic variables ($U^{(i)} = \mathrm{m}_i \dot{u}^{(i)}$ is the momentum conjugated to $u^{(i)}$) and the phase space is the "collisionless" open domain in $\mathbb{R}^{6(n+1)}$ given by

$$\mathcal{M} := \{U^{(i)}, u^{(i)} \in \mathbb{R}^3 : u^{(i)} \ne u^{(j)}, 0 \le i \ne j \le n\}$$

endowed with the standard symplectic form

$$\sum_{i=0}^{n} dU^{(i)} \wedge du^{(i)} := \sum_{\substack{0\le i\le n \\ 1\le k\le 3}} dU_k^{(i)} \wedge du_k^{(i)}. \tag{3}$$

We recall that the Hamiltonian equations associated to a Hamiltonian function $H(p,q) = H(p_1,...,p_n,q_1,...,q_n)$, where (p,q) are standard symplectic variables (i.e., the associated symplectic form is $dp \wedge dq = \sum_{i=1}^{n} dp_i \wedge dq_i$) are given by

$$\begin{cases} \dot{p} = -\partial_q H \\ \dot{q} = \partial_p H \end{cases} \quad \text{i.e.,} \quad \begin{cases} \dot{p}_i = -\partial_{q_i} H, \\ \dot{q}_i = \partial_{p_i} H, \quad (1 \le i \le n). \end{cases} \tag{4}$$

We shall denote the standard Hamiltonian flow, namely, the solution of (4) with initial data p_0 and q_0, by $\phi_H^t(p_0,q_0)$. For general information, see (Arnold et al, 2006).

2.2 The Linear Momentum Reduction

In view of the invariance properties discussed above, it is enough to consider the submanifold

$$\mathcal{M}_0 := \left\{ (U,u) \in \mathcal{M} : \sum_{i=0}^{n} \mathrm{m}_i u^{(i)} = 0 = \sum_{i=0}^{n} U^{(i)} \right\},$$

which corresponds to the manifold described in (2).

The submanifold $\mathcal{M}_0$ is symplectic, i.e., the restriction of the form (3) to $\mathcal{M}_0$ is again a symplectic form; indeed:

$$\left(\sum_{i=0}^{n} dU^{(i)} \wedge du^{(i)} \right) \Bigg|_{\mathcal{M}_0} = \sum_{i=1}^{n} \frac{\mathrm{m}_0 + \mathrm{m}_i}{\mathrm{m}_0} dU^{(i)} \wedge du^{(i)}.$$

Following Poincaré, one can perform a *symplectic reduction* (“reduction of the linear momentum”) allowing to lower the number of degrees of freedom by three units; recall that the number of degree of freedom of an autonomous Hamiltonian system is half of the dimension of the phase space (classically, the dimension of the configuration space). Indeed, let $\phi_{\mathrm{he}} : (R,r) \to (U,u)$ be the linear transformation given by

$$\phi_{\mathrm{he}} : \begin{cases} u^{(0)} = r^{(0)}, \qquad u^{(i)} = r^{(0)} + r^{(i)}, \quad (i = 1,\dots,n) \\ U^{(0)} = R^{(0)} - \sum_{i=1}^{n} R^{(i)}, \quad U^{(i)} = R^{(i)}, \qquad (i = 1,\dots,n); \end{cases} \tag{5}$$

such transformation is symplectic, i.e.,

$$\sum_{i=0}^{n} dU^{(i)} \wedge du^{(i)} = \sum_{i=0}^{n} dR^{(i)} \wedge dr^{(i)};$$

recall that this means, in particular, that in the new variables the Hamiltonian flow is again standard: more precisely, one has that $\phi^t_{\mathcal{H}_{\rm N}} \circ \phi_{\rm he} = \phi_{\rm he} \circ \phi^t_{\mathcal{H}_{\rm N}\circ\phi}$, where the subscript "he" signifies helium (sun) and the little circles mean composition.

Letting

$$\mathrm{m}_{\rm tot} := \sum_{i=0}^{n} \mathrm{m}_i$$

one sees that, in the new variables, $\mathcal{M}_0$ reads

$$\{(R,r) \in \mathbb{R}^{6(n+1)} : R^{(0)} = 0, r^{(0)} = -\mathrm{m}_{\rm tot}^{-1} \sum_{i=1}^{n} \mathrm{m}_i r^{(i)} \text{ and } 0 \neq r^{(i)} \neq r^{(j)}\ \forall 1 \leq i \neq j \leq n\}.$$

The restriction of the 2–form (3) to $\mathcal{M}_0$ is simply $\sum_{i=1}^{n} dR^{(i)} \wedge dr^{(i)}$ and

$$\left(\mathcal{H}_{\rm N} \circ \phi_{\rm he}\right)|_{\mathcal{M}_0} = \sum_{i=1}^{n} \left(\frac{|R^{(i)}|^2}{2\,\frac{\mathrm{m}_0 \mathrm{m}_i}{\mathrm{m}_0+\mathrm{m}_i}} - \frac{\mathrm{m}_0 \mathrm{m}_i}{|r^{(i)}|} \right) + \sum_{1\leq i<j\leq n} \left(\frac{R^{(i)} \cdot R^{(j)}}{\mathrm{m}_0} - \frac{\mathrm{m}_i \mathrm{m}_j}{|r^{(i)} - r^{(j)}|} \right)$$
$$=: \mathcal{H}_{\rm N}.$$

Thus, the dynamics generated by $\mathcal{H}_{\rm N}$ on $\mathcal{M}_0$ is equivalent to the dynamics generated by the Hamiltonian $(R,r) \in \mathbb{R}^{6n} \to \mathcal{H}_{\rm N}(R,r)$ on

$$\mathcal{M}_0 := \{(R,r) = (R^{(1)},...,R^{(n)}, r^{(1)},...,r^{(n)}) \in \mathbb{R}^{6n} : 0 \neq r^{(i)} \neq r^{(j)}\ \forall 1 \leq i \neq j \leq n\}$$

with respect to the standard symplectic form $\sum_{i=1}^{n} dR^{(i)} \wedge dr^{(i)}$; to recover the full dynamics on $\mathcal{M}_0$ from the dynamics on $\mathcal{M}_0$ one will simply set $R^{(0)}(t) \equiv 0$ and $r^{(0)}(t) := -\mathrm{m}_{\rm tot}^{-1} \sum_{i=1}^{n} \mathrm{m}_i r^{(i)}(t)$.

Since we are interested in the planetary case, we perform the trivial rescaling by a small positive parameter μ:

$$m_0 := \mathrm{m}_0, \mathrm{m}_i = \mu m_i \quad (i \geq 1), \quad X^{(i)} := \frac{R^{(i)}}{\mu}, \quad x^{(i)} := r^{(i)},$$

$$\mathcal{H}_{\rm plt}(X,x) := \frac{1}{\mu} \mathcal{H}_{\rm N}(\mu X, x),$$

which leaves unchanged Hamilton's equations. Explicitly, if

$$M_i := \frac{m_0 m_i}{m_0 + \mu m_i}, \qquad \text{and} \qquad \bar{m}_i := m_0 + \mu m_i,$$

then

$$\begin{aligned}\mathcal{H}_{\text{plt}}(\mathrm{X},\mathrm{x}) &:= \sum_{i=1}^{n}\left(\frac{|\mathrm{X}^{(i)}|^2}{2M_i} - \frac{M_i \bar{m}_i}{|x^{(i)}|}\right) + \mu \sum_{1\le i<j\le n}\left(\frac{X^{(i)}\cdot X^{(j)}}{m_0} - \frac{m_i m_j}{|x^{(\mathrm{i})}\text{-}x^{(j)}|}\right) \\ &=: \mathcal{H}_{\text{plt}}^{(0)}(X,x) + \mu \mathcal{H}_{\text{plt}}^{(1)}(X,x),\end{aligned} \tag{6}$$

the phase space being

$$\mathcal{M} := \{(X,x) = (X^{(1)},\ldots,X^{(n)},x^{(1)},\ldots,x^{(n)}) \in \mathbb{R}^{6n} : 0 \neq x^{(i)} \neq x^{(j)} \quad \forall 1 \le i \neq j \le n\},$$

endowed with the standard symplectic form $\sum_{i=1}^{n} dX^{(i)} \wedge dx^{(i)}$.

Recall that $F(X,x)$ is an integral for $\mathcal{H}(X,x)$ if $\{F,\mathcal{H}\}=0$ where $\{F,G\} = F_X \cdot G_x - F_x \cdot G_X$ denotes the (standard) Poisson bracket. Now, observe that while $\sum_{i=1}^{n} X^{(i)}$ is obviously *not* an integral for $\mathcal{H}_{\text{plt}}$, the transformation (5) does preserve the total *angular momentum* $\sum_{i=0}^{n} U^{(i)} \times u^{(i)}$, "$\times$" denoting the standard vector product in $\mathbb{R}^3$, so that the total angular momentum

$$\mathrm{C}=(\mathrm{C}_1,\mathrm{C}_2,\mathrm{C}_3) := \sum_{i=1}^{n} \mathrm{C}^{(i)}, \qquad \mathrm{C}^{(i)} := X^{(i)} \times x^{(i)}, \tag{7}$$

is still a (vector–valued) integral for $\mathcal{H}_{\text{plt}}$. The integrals C_i, however, do not commute (i.e., their Poisson brackets do not vanish):

$$\{\mathrm{C}_1,\mathrm{C}_2\} = \mathrm{C}_3, \qquad \{\mathrm{C}_2,\mathrm{C}_3\} = \mathrm{C}_1, \qquad \{\mathrm{C}_3,\mathrm{C}_1\} = \mathrm{C}_2,$$

but, for example, $|\mathrm{C}|^2$ and C_3 are two commuting, independent integrals.

2.3. Delaunay Variables

The Hamiltonian $\mathcal{H}_{\text{plt}}^{(0)}$ in (6) governs the motion of n decoupled two–body (signified by the subscript 2B) problems with Hamiltonian

$$h_{2\mathrm{B}}^{(i)} = \frac{|X^{(i)}|^2}{2M_i} - \frac{M_i \bar{m}_i}{|x^{(i)}|}, \qquad (X^{(i)},x^{(i)}) \in \mathbb{R}^3 \times \mathbb{R}^3_* := \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}).$$

Such two–body systems are, as well known, integrable. The explicit "symplectic integration" is done by means of the *Delaunay variables,* whose construction we, now, briefly, recall (for full details and proofs, see, e.g., (Celletti and Chierchia, 2007)).

Assume that $h^{(i)}_{2B}\left(X^{(i)},x^{(i)}\right)<0$ so that the Hamiltonian flow $\phi^t_{h^{(i)}_{2B}}\left(X^{(i)},x^{(i)}\right)$ evolves on a Keplerian ellipse $\mathfrak{E}_i$ and assume that the eccentricity $e_i \in (0,1)$.

Let a_i, P_i denote, respectively, the *semimajor axis* and the *perihelion* of $\mathfrak{E}_i$.

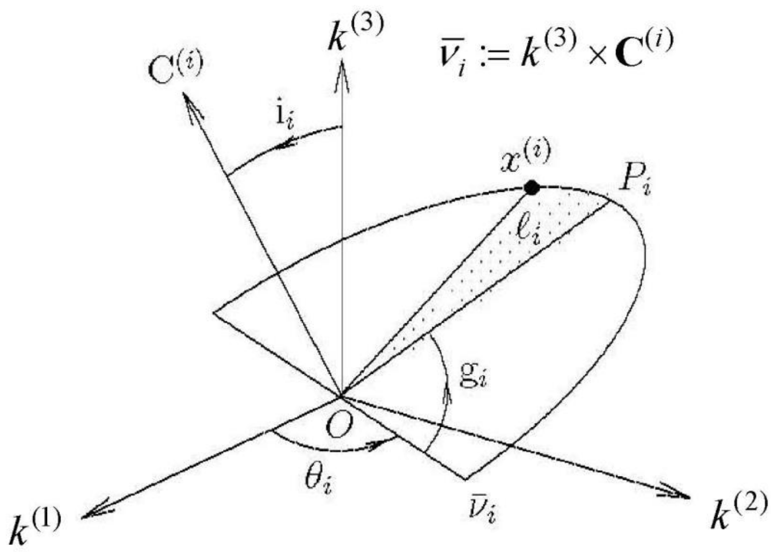

Figure 1. Spatial Delaunay angle variables.

Let us, also, introduce the "Delaunay nodes"

$$\bar{\nu}_i := k^{(3)} \times \mathrm{C}^{(i)} \quad 1 \le i \le n, \tag{8}$$

where $(k^{(1)},k^{(2)},k^{(3)})$ is the standard orthonormal basis in $\mathbb{R}^3$. Finally, for $u,v \in \mathbb{R}^3$ lying in the plane orthogonal to a non–vanishing vector w, let $\alpha_w(u,v)$ denote the positively oriented angle (mod 2π) between u and v (orientation follows the "right hand rule").

The *Delaunay action–angle variables* $\left(\Lambda_i,\Gamma_i,\Theta_i,\ell_i,g_i,\theta_i\right)$ are, then, defined as

$$\begin{cases}\Lambda_i := M_i\sqrt{\bar{m}_i a_i}\\ \ell_i := \text{mean anomaly of } x^{(i)} \text{ on } \mathfrak{E}_i\end{cases}$$

$$\begin{cases}\Gamma_i := |\mathrm{C}^{(i)}| = \Lambda_i\sqrt{1-e_i^2} \\ \mathrm{g}_i := \alpha_{\mathrm{C}^{(i)}}(\bar{\nu}_i, P_i)\end{cases} \tag{9}$$

$$\begin{cases}\Theta_i := \mathrm{C}^{(i)}\cdot k^{(3)} \\ \theta_i := \alpha_{k^{(3)}}(k^{(1)}, \bar{\nu}_i)\end{cases}$$

Notice that the Delaunay variables are defined on an open set of full measure of the Cartesian phase space $\mathbb{R}^{3n}\times\mathbb{R}^{3n}_*$, namely, on the set where $e_i \in (0,1)$ and the nodes $\bar{\nu}_i$ in (8) are well defined; on such set the "Delaunay inclinations" i_i defined through the relations

$$\cos \mathrm{i}_i := \frac{\mathrm{C}^{(i)}\cdot k^{(3)}}{|\mathrm{C}^{(i)}|} = \frac{\Theta_i}{\Gamma_i}, \tag{10}$$

are well defined and we choose the branch of $\cos^{-1}$ so that $\mathrm{i}_i \in (0,\pi)$.

The Delaunay variables become singular when $\mathrm{C}^{(i)}$ is vertical (the Delaunay node is no more defined) and in the circular limit (the perihelion is not unique). In these cases different variables have to be used (see below).

On the set where the Delaunay variables are well posed, they define a *symplectic set of action–angle variables*, meaning that

$$\sum_{i=1}^{n} dX^{(i)}\wedge dx^{(i)} = \sum_{i=1}^{n} d\Lambda_i\wedge d\ell_i + d\Gamma_i\wedge dg_i + d\Theta_i\wedge d\theta_i;$$

for a proof, see Section 3.2 of (Celletti and Chierchia, 2007).

In Delaunay action–angle variables $((\Lambda,\Gamma,\Theta),(\ell,\mathrm{g},\theta))$ the Hamiltonian $\mathcal{H}^{(0)}_{\mathrm{plt}}$ takes the form

$$-\sum_{i=1}^{n}\frac{M_i^3\bar{m}_i^2}{2\Lambda_i^2} =: h_{\mathrm{K}}(\Lambda). \tag{11}$$

We shall restrict our attention to the collisionless phase space

$$\mathcal{M}_{\mathrm{plt}} := \{(\Lambda,\Gamma,\Theta)\in\mathbb{R}^{3n}: \quad \Lambda_i > \Gamma_i > \Theta_i > 0, \quad \frac{\Lambda_i}{M_i\sqrt{\bar{m}_i}} \neq \frac{\Lambda_j}{M_j\sqrt{\bar{m}_j}}, \forall i\neq j\}\times\mathbb{T}^{3n},$$

endowed with the standard symplectic form

$$\sum_{i=1}^{n} d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge dg_i + d\Theta_i \wedge d\theta_i.$$

Notice that the $6n$–dimensional phase space $\mathcal{M}_{\rm plt}$ is foliated by $3n$–dimensional $\mathcal{H}^{(0)}_{\rm plt}$–invariant tori $\{\Lambda, \Gamma, \Theta\} \times \mathbb{T}^3$, which, in turn, are foliated by n–dimensional tori $\{\Lambda\} \times \mathbb{T}^n$, expressing geometrically the degeneracy of the integrable Keplerian limit of the $(1+n)$–body problem.

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Biographical Sketch

Luigi Chierchia was born in 1957 in Rome, Italy. He received his PhD degree from Courant Institute in 1986. He has been a full professor in Mathematical Analysis since 2002 at the Mathematics Department of Roma Tre University. He has contributed to nonlinear differential equations and dynamical systems with emphasis on stability problems in Hamiltonian systems. Prix 1995 Institut Henri Poincaré (first edition). Invited speaker at the 2014 International Congress of Mathematicians. The main-belt asteroid *Chierchia-2003 OC21* has been given his name.