STOCHASTIC DIFFERENTIAL EQUATIONS

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Contents

Summary

The motion of a particle in a liquid can be described by a stochastic differential equation. The most important question is the existence and unicity of the solution of such an equation.

1. Existence and Unicity

The process $X(t)$ defined in the article *Stochastic Calculus* describes the motion of a particle when there is no macroscopic velocity of the liquid in which the particle moves. Now we consider the situation when the liquid is not homogeneous and not motionless.

Let $m(x,t)$ be the macroscopic velocity of a small volume V of a liquid located at xR^1 at time t. Now the motion of a particle in a time-interval $(t,t + dt)$ arises from two sources: the macroscopic motion of the liquid which is $m(X(t),t)dt$ (where $X(t)$ is the location of the underlying particle at time (t) and the microscopic influence of the liquid which is:

$$\sigma(X(t),t)(W(t+dt)-W(t)).$$

Hence we get the "stochastic differential equation":

$$X'(t)=m(X(t),t)+\sigma(X(t),t)W'(t).$$

Since $W'(t)$ does not exist, precisely speaking the above equation is meaningless.

Therefore instead of it we consider the integral equation:

$$X(t)-X(a)=\int_a^t m(X(s),s)ds+\int_a^t \sigma(X(s),s)dW(s). \qquad (1)$$

Now we say that the motion of a particle is described by a stochastic process $X(t)$ which satisfies the above integral equation and an initial condition $X(a) = X$ where X is a random variable independent from $\{W(t), a \le t \le b\}$. Clearly we have to give conditions that imply the existence and uniqueness of such a process. In fact we assume that the functions m and σ are regular enough. Namely:

The functions m and σ are Borel measurables that satisfy (for some $k > 0$) the Lipschitz condition

$$|m(x,t) - m(y,t)| \le k|x - y|,$$

$$|\sigma(x,t) - \sigma(y,t)| \le k|x - y|$$

for all x,y,t. We also assume that:

$$|m(x,t)| \le k(1 + x^2)^{1/2},$$

$$|\sigma(x,t)| \le k(1 + x^2)^{1/2}.$$

Assuming the above conditions we have the following:

Theorem: *The integral equation* (1) *has one and only one solution* $X(t)$ *which satisfies the initial condition* $X(a) = X$. *We also have:*

$$\int_a^b \mathbf{E} X^2(t)dt < \infty,$$

$X(t)$ *is continuous on* $[a,b]$ *with probability one and* $\{X(t), a \le t \le b\}$ *is a Markov process*.

We note that the proof of this theorem is based on a successive approximation procedure. In fact let

$$X_0(t) \equiv X \, (a \le t \le b)$$

and

$$X_{n+1}(t) = X + \int_a^t m(X_n(s),s)ds + \int_a^t \sigma(X_n(s),s)dW(s).$$

Then we have to prove that the above defined sequence $\{X_n(t)\}$ converges a.s. (as $n \to \infty$) to a stochastic process that is a solution of the underlying stochastic differential equation and that satisfies the other statements of the above theorem as well.

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Biographical Sketch

P. Révész was born in 1934. He gained his Ph.D. in Budapest in 1958, Budapest. He was Associate Professor at the University of Budapest, 1957–1964, a Fellow of the Mathematical Institute of the Hungarian Academy of Sciences, 1964–1984, and Professor at the Vienna University of Technology from 1985 until his retirement in1998. He is a member of the Hungarian Academy of Sciences, Academia Europaea, International Statistical Institute, Institute of Mathematical Statistics, and the Bernoulli Society (of which he was President, 1983–1985). His publications include *The Laws of Large Numbers* (Academic Press, New York 1967), *Strong Approximations in Probability and Statistics* (Academic Press, New York 1981, co-authored with M. Csörgõ), *Random Walk in Random and Non-Random Environments* (World Scientific, Singapore 1990), and *Random Walks of Infinitely Many Particles* (World Scientific, Singapore 1994).