THE EQUIVALENCE PRINCIPLE

Birgit Grodal
University of Copenhagen, Denmark

Keywords: equivalence, Walrasian equilibrium, Walrasian allocation, atomless economy, large finite economy, strongly fair net trades, core, Mas-Colell Bargaining set, Geanakoplos bargaining set, value, approximately decentralization, Nash equilibrium.

Contents

Summary

In this chapter we survey some equivalence results. The starting point is the set of Walrasian allocations. We first show that a Walrasian allocation can be characterized by the property that it has strongly fair net trades. Then we consider atomless economies. An atomless economy formalizes the assumption that the economy consists of many small agents. In an atomless economy we present the Core, the Bargaining Set, and Value equivalence results.

We also examine large finite economies. We present a Core decentralization result and also a decentralization result for the Geanakoplos Bargaining Set. The Mas-Colell Bargaining Set does not lead to a convergence result in large finite economies.

Finally, we give a few examples of equivalence between the set of Walrasian equilibria

in a finite economy and the set of Nash equilibria in suitably defined non-cooperative games.

1. Introduction

The starting point of this survey is a pure exchange economy with finitely many commodities and with private ownership of the initial endowments. In such an economy it is often assumed that a Walrasian market gives the trading possibilities for the consumers. A Walrasian market is the institution given by a price system. All consumers take the prices of the commodities as parametrically given and choose an optimal action given these prices. The prices defining the Walrasian market are set such that aggregate demands equal aggregate supplies. Much of economic theory is devoted to analyzing economies with Walrasian markets or variants of this model.

However, considering an economy with a Walrasian market does not justify the Walrasian institution. How can it be justified that the trading possibilities for the agents are defined by a price system and that agents take the price system as parametrically given? Game theory has been extremely useful in the search for an answer to this question.

Concepts from cooperative as well as non-cooperative game theory have been used to introduce new equilibrium concepts into economics. These equilibrium notions do not rest on the assumption that agents take the prices of commodities as given. Thus, one has been able to ask the question, whether some of these other equilibrium notions lead to an equivalence result in the following sense: An allocation of the commodities to the agents in the economy is an equilibrium state according to this new equilibrium concept if and only if there exists a price system p such that the allocation is an equilibrium allocation corresponding to the Walrasian market defined by the price system p. If an equivalence result obtains we have an endogenous explanation of the Walrasian institution.

For most of the equilibrium concepts used in game theory there is no assumption paralleling the assumption that the agents take the Walrasian market as given *a priori*. Clearly, if prices are always set such that demands equal supplies, then in a finite economy any agent shall be able to influence the price system. However, the implicit assumption is that agents behave as if their actions have no affect on the price system. Clearly, one may think, that if the economy consists of many small agents who act independently, then this implicit assumption is approximately satisfied. Aumann (1964) defined a continuum economy in which the agents were modeled as [0, 1] with the Lebesgue measure. In Aumann's model the assumption that an individual agent cannot influence the price system is endogenous and Aumann gave the first general equivalence theorem. He proves that an allocation can be obtained via a Walrasian market if and only if there is no group of consumers, which by using its own initial endowments can ensure that all its members are better of. This is Aumann's classical Core equivalence theorem.

Since Aumann's result, many other equivalence results have been obtained for economies with an atomless measure space of consumers. These results have very much enlarged our understanding of the foundation for the Walrasian market institution. Moreover, the attempts to analyze economies with infinitely many commodities have given new insights. Ostroy and Zame (1994) have pointed out that, when the commodity space is infinitely dimensional, an atomless measure space of agents is, in general, not enough to obtain results analogous with the equivalence results for economies with finitely many commodities.

Clearly, modeling the agents in an economy as an atomless measure space is an abstraction. Hence, a fundamental question is whether the equivalence results for atomless economies have analogies in economies with large, but finite numbers of agents. A strong result in this direction is the classical theorem by Debreu and Scarf (1963). They showed that the Core and set of Walrasian allocations become arbitrarily close when a finite economy is replicated sufficiently many times. However, Bewley (1973) showed that if one considers more general sequences of finite economies, one cannot, in general, hope for such a strong conclusion. This leads to a weaker question. Namely, whether for some of the game theoretical solution concepts, one will have that any equilibrium allocation can be approximately decentralized by a Walrasian market in large finite economies.

Searching for equivalence results has a parallel in classical welfare economics. For a long time, it has been known that allocations obtained via a Walrasian market are Pareto efficient. However, starting with a Pareto efficient allocation, a transfer of initial endowments among the agents is necessary if the allocation has to be obtained from a Walrasian market. This is the content of the classical First and Second welfare theorems, see Debreu (1959).

2. Notation and the Basic Model

For two vectors $x, y \in \mathbb{R}^\ell$, we use the notation $y \geq x$ if $y_h \geq x_h$ for all $h = 1, ..., \ell$; $y > x$ if $y_h \geq x_h$ for all $h = 1, ..., \ell$ and $y \neq x$; and $y >> x$ if $y_h > x_h$ for all $h = 1, ..., \ell$. We let $\Delta = \{p \in \mathbb{R}^\ell_+ \mid \sum p_h = 1\}$ be the non-negative price simplex in $\mathbb{R}^\ell$. For a set S let $|S|$ denote the number of elements in S. $\mathbf{Z}_+$ denotes the non-negative integers. For $x \in \mathbb{R}$ we let $\|x\|$ denote the Euclidean norm of x.

We consider economies in which all consumers have the positive orthant $\mathbb{R}^\ell_+$ as *consumption sets*.

A *preference relation* $\succ$ on $\mathbb{R}^\ell_+$ is said to be *continuous* if the set $\{(x, y) \in \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+ \mid y \succ x\}$ is open relative to $\mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+$. The relation $\succ$ is irreflexive if $x \not\succ x$ for all $x \in \mathbb{R}^\ell_+$. It is *monotonic* if for all $x, y \in \mathbb{R}^\ell_+$ with $y > x$ we have $y \succ x$. A preference relation on $\mathbb{R}^\ell_+$ is said to be *transitive-monotonic* if $z \geq y$ and $y \succ x$ imply $z \succ x$ for all $x, y, z \in \mathbb{R}^\ell_+$. We let $\mathcal{P}_{mo}$ be the set of continuous, irreflexive, monotonic, and transitive-monotonic preference relations on $\mathbb{R}^\ell_+$. A preference relation $\succsim$ on $\mathbb{R}^\ell_+$ is complete if y

$\succsim$ x or $x \succsim y$ for all $x, y \in \mathbb{R}^{\ell}_{+}$. The relation $\succsim$ is transitive if $z \succsim y$ and $y \succsim x$ imply $z \succsim x$ for all $x, y, z \in \mathbb{R}^{\ell}_{+}$. We say that $\succ \in \mathcal{P}_{mo}$ is derived from the complete and transitive preference relation $\succsim$ when $y \succ x$ if and only if $y \succsim x$ and $x \not\succsim y$. We let $\mathcal{P}^{*}_{mo} = \{\succ \in \mathcal{P}_{mo} \mid \succ$ is derived from a complete and transitive relation $\succsim\}$. A preference relation $\succ \in \mathcal{P}^{*}_{mo}$ is said to be *smooth* if the corresponding preference relation $\succsim$ can be represented by a strictly quasiconcave C^2 utility function $u : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ with positive Gaussian curvature u^c. (The function u is strictly quasiconcave, if $u(\lambda x + (1 - \lambda)y) > \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}^{\ell}_{+}$, $x \neq y$, and $\lambda \in (0, 1)$.)

A pure *exchange economy* with private ownership is a mapping $\mathcal{E} : (A, \mathcal{A}, \lambda) \to \mathbb{R}^{\ell}_{+} \times \mathcal{P}_{mo}$, where $a \mapsto \mathcal{E}(a) = (e(a), \succ_a)$. $\mathcal{A}$ is a σ-field of subsets of A. λ is a finite non-negative measure on $\mathcal{A}$. A is the set of consumers. An element $S \in \mathcal{A}$ is a *coalition* of consumers. A coalition S is said to be *non-null* if $\lambda(S) > 0$. We shall assume that the measure space is complete. Thus, all sets $S \subset A$ for which there exists a null set $T \in$ where $S \subset T$ are again in . The vector $e(a)$ is the initial endowment of consumer a and $\in$ is consumer a's preference relation on . We assume that $e : A \to$ is an integrable function with . Furthermore, we assume that is measurable in the sense that for any measurable functions $f, g : A \to$ we have $\{a \in A \mid f(a)\ g(a)\} \in$.

Consider a consumer a in the economy and a consumption plan $x \in$. Then we define a's *net trade* as $x - e(a)$. Since we have assumed that the consumers' consumption sets equal then the set of net trades which are individually feasible for a is $- \{e(a)\}$.

Definition 1

Let be an economy. An allocation *for the coalition S is an integrable function* $x : S \to$. *An* attainable allocation x for the coalition $S \in$ *is an allocation for S such that*

$x(a) \in$ *for a.a.* $a \in A$ *and* .

An attainable allocation x *is an allocation which is attainable for A. We let* $X()$ *denote the allocations that are attainable in the economy* .

Thus, an allocation x is attainable for the coalition S if S can ensure its members $x(a)$, $a \in S$, by using its aggregate initial endowment .

An allocation $x \in X(\mathcal{E})$ is said to be *individually rational* if $e(a) \not\succ_a x(a)$ for a.a. $a \in A$. Thus, an allocation x is individually rational if there is no coalition with positive measure such that all agents in the coalition prefer their initial endowments to the bundle they obtain by x. An allocation $x \in X(\mathcal{E})$ is said to be *Pareto efficient* if there does not exist $y \in X(\mathcal{E})$ such that $y(a) \succ_a x(a)$ for a.a. $a \in A$. Thus an allocation x is Pareto efficient if it is impossible to distribute the total initial endowments in the economy such that almost all agents in A get bundles they prefer to the bundles obtained by x. When the consumers in an economy $\mathcal{E}$ have preferences in $\mathcal{P}_{mo}^{*}$, then we say that an allocation has *equal treatment* if $x(a) \sim_a x(b)$ for almost all $a, b \in A$ for which $(e(a), \succ_a) = (e(b), \succ_b)$.

2.1. Atomless Economies

An economy $\mathcal{E} : (A, \mathcal{A}, \lambda) \to \mathbb{R}_+^{\ell} \times \mathcal{P}_{mo}$ is called an *atomless economy* if $(A, \mathcal{A}, \lambda)$ is an atomless measure space. That is, for all $S \in \mathcal{A}$ with $\lambda(S) > 0$ there exists $B \subset S$, $B \in \mathcal{A}$ such that $\lambda(B) > 0$ and $\lambda(S \setminus B) > 0$. Hence, an economy is atomless if any non-null coalition can be split into two non-null coalitions. Clearly, if an economy is atomless then each individual agent is a null set and there is necessarily a more than countable number of agents in the economy. Atomless economies were introduced by Aumann (1964) as a way to formalize that the economy consists of many (a continuum of) small agents. Modeling a real world economy as an atomless economy makes it endogenous that agents individually have no influence on the set of attainable allocations for any coalition. If an allocation is attainable for S and a null set of agents changes their consumption plan, the new allocation is again attainable for S.

A useful tool in analyzing atomless economies is Lyapunov's Theorem as introduced into economics by Vind (1964).

Theorem 1 **(Lyapunov)**

Consider a finite family of finite non-negative atomless measures $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ on the measurable space $(A, \mathcal{A})$. Then the range $R(\mu) = \{x \in \mathbb{R}^n \mid$ there exists $C \in \mathcal{A}$ where $x_h = \mu_h(C)$, $h = 1, \ldots, n\}$ is a compact and convex subset of $\mathbb{R}^n$.

Clearly, Lyapunov's Theorem implies that for an atomless economy $\mathcal{E}$ with consumers in $(A, \mathcal{A}, \lambda)$ and an integrable function $x : A \to \mathbb{R}^{\ell}$, $\{\int_S x d\lambda \mid S \in \mathcal{A}\}$ is a convex subset of $\mathbb{R}^{\ell}$. Moreover for any correspondence (set-valued function) $\phi : A \Rightarrow \mathbb{R}^{\ell}$, the set $\int_A \phi d\lambda = \{\int_A f d\lambda \mid f(a) \in \phi(a)$ a.a. $a \in A$ and f integrable$\}$ is convex.

2.2. Finite Economies

A finite economy is an economy $\mathcal{E} : (A, \mathcal{A}, \lambda) \to \mathbb{R}_+^{\ell} \times \mathcal{P}_{mo}$ where A is a finite set, $\mathcal{A}$ is all subsets of A, and λ is the counting measure, that is, $\lambda S) = \frac{|S|}{|A|}$ for all $S \subset A$.

A useful tool in analyzing large finite economies is the Shapley-Folkman Theorem as introduced by Starr (1969).

Theorem 2 **(Shapley-Folkman)**

Let Z_i, $i = 1, ..., n$ *be a family of non-empty subsets of* $\mathbb{R}^\ell$ *and let* $u \in conv \sum_{i=1}^{n} Z_i$. *Then there are points* $u_i \in \text{conv}Z_i$, $i = 1, ..., n$, *such that* $x = \sum_{i=1}^{n} u_i$ *with* $u_i \in Z_i$ *except for at most* ℓ *of the points.*

Note in particular, that the number of exceptional points, that is, points which are not in Z_i, depends on the dimension ℓ of the Euclidean space but not on the number of sets in the family. The Shapley-Folkman Theorem is an approximate version of Lyapunov's Theorem. Consider for example the case where the sets Z_i, $i = 1, ..., n$, are uniformly bounded. Then the Euclidean distance between the convex hull of the sum of the sets and the sum itself is bounded independently of the number of sets in the family.

3. Walrasian Equilibrium

3.1. Walrasian Allocations

We shall now define the set of allocations, which can be obtained by the Walrasian institution. That is, attainable allocations that can be obtained by letting each consumer independently choose an optimal net trade in a *Walrasian market* $M(p) = \{z \in \mathbb{R}^\ell \mid p \cdot z \leq 0\}$.

Definition 2

Let $\mathcal{E}$ be an economy. The pair $(p, x) \in \mathbb{R}^\ell \backslash \{0\} \times X(\mathcal{E})$ consisting of a price system and an attainable allocation is a Walrasian Equilibrium for $\mathcal{E}$ if [(i)]$p(x(a) - e(a)) \leq 0$ for a.a. $a \in A$, $y \succ_a x(a) \Rightarrow p(y - e(a)) > 0$ a.a. $a \in A$.

A Walrasian allocation *is an allocation x for which there exists a price system p such that* (p, x) *is a Walrasian Equilibrium. We let* $W(\mathcal{E})$ *denote the set of Walrasian allocations for the economy* $\mathcal{E}$.

In a Walrasian equilibrium all consumers take the Walrasian market $M(p) = \{z \in \mathbb{R}^\ell \mid p \cdot z \leq 0\}$ with the price system p as given and choose net trades so as to maximize their preference relations. If the economy $\mathcal{E}$ is atomless then of course no agent will be able to manipulate the Walrasian price system. More precisely, assume that prices are set such that markets clear. Then the price system clears the markets independent of the action of an individual agent (and a null set of agents).

3.2. Strongly Fair Net Trades

An elementary characterization of a Walrasian allocation for a finite economy $\mathcal{E}$ is given in Schmeidler and Vind (1972).

Definition 3

Let $\mathcal{E}$ *be a finite economy. The allocation x has* strongly fair net trades *if for all agents* $a \in A$ *and all* $n_b \in \mathbf{Z}_+$

$$\sum_{b \in A} n_b(x(b) - e(b)) + e(a) \in \mathbb{R}^{\ell}_{+} \Rightarrow \sum_{b \in A} n_b(x(b) - e(b)) + e(a) \not\succ_a x(a).$$

The idea behind the concept of strongly fair net trades is the following: Each agent a considers the net trades obtained by the agents in A, that is the set $Zx = \{x(b) - e(b) \in \mathbb{R}^{\ell} \mid b \in A\}$ of net trades revealed by x. If the institution leading to x is fair, then all the net trades in Z_x should be available to any of the consumers. Hence, in equilibrium, none of the consumers should prefer any of these net trades to the net trade they themselves have obtained. (This equilibrium condition leads to the concept of allocations having *fair net trades*.) However, one might argue that an agent should also be able to obtain a net trade which is the sum of net trades revealed by x, and also any net trade which is a linear combination of such net trades with non-negative integer weights. An agent just uses the market possibilities repeatedly. In equilibrium no agent should prefer such a combination of the net trades revealed by x. This is exactly what the condition in the definition of strongly fair net trades says.

Clearly, any Walrasian allocation has strongly fair net trades. Schmeidler and Vind (1972) show that apart from indivisibilities, this condition also characterizes a Walrasian allocation in the following sense. Assume that $X \subset \mathbb{R}^{\ell}$ is the marketed subset of the commodity space, that is, for any price system $p \in \mathbb{R}^{\ell}$ the Walrasian market given X equals $\{z \in X \mid p \cdot z \le 0\}$. Thus for any price system p the consumers cannot choose net trades in the whole of $\mathbb{R}^{\ell}$ but only in the marketed space X. We can now define the set of Walrasian allocations relative to X. The definition of a Walrasian allocation above being the special case where $X = \mathbb{R}^{\ell}$. Vind and Schmeidler show that if the attainable allocation x has strongly fair net trades and reveals divisibility (for a precise definition see Schmeidler and Vind) then x is a Walrasian allocation relative to the smallest linear subspace of $\mathbb{R}^{\ell}$ containing $\{(x(a) - e(a)) \mid a \in A\} \cup \{c\}$ for any $c \in \mathbb{R}^{\ell}$, $c >> 0$. In particular, if the dimension of smallest linear subspace containing $\{(x(a) - e(a)) \mid a \in A\}$ has dimension $\ell - 1$, then x is a Walrasian allocation.

The main insight used in the proof of Schmeidler and Vind's theorem is that when x is an attainable allocation, then the set $\tilde{Z}_x = \{ \sum_{b \in A} n_b(x(b) - e(b)) \mid n_b \in \mathbf{Z}_+\}$ with addition is a group. Clearly $\tilde{Z}_x$ is closed under addition and $0 \in \tilde{Z}_x$. To see that all $z \in \tilde{Z}_x$ have inverse elements in $\tilde{Z}_x$ consider any $z = \sum_{b \in A} n_b(x(b) - e(b)) \in \tilde{Z}_x$. Since $x(b) - e(b) = -\sum_{a \neq b}(x(a) - e(a))$ for all $b \in A$, then $-z$ is also in $\tilde{Z}_x$.

A theorem corresponding to Schmeidler and Vind's also holds for an atomless economy $\mathcal{E}$. Define for each attainable allocation x the net trade set $\bar{Z}_x = \{ \int_S (x - e)\, d\lambda \mid S \in \mathcal{A} \}$. We now say that the allocation x has strongly fair net trades if, for no non-null coalition S, there exists an integrable function $y : S \to \mathbb{R}^{\ell}_{+}$ such that

[(i)] for all $S' \subset S$, $S' \in \mathcal{A}$, $\int_{S'} (y - e)\, d\lambda \in \bar{Z}_x$, and $y(a) \succ_a x(a)$ for a.a. $a \in S$.

It is easily seen that a Walrasian allocation has still strongly fair net trades. The opposite conclusion, namely that an attainable allocation *x* with strongly fair net trades is a Walrasian allocation relative to the smallest linear subspace of $\mathbb{R}^\ell$ containing $\bar{Z}_x \cup \{c\}$ for any $c \in \mathbb{R}^\ell$, $c >> 0$, also holds true. This follows, as in Schmeidler and Vind's theorem, since the set $\bar{Z}_x$ is compact and convex by Lyapunov's Theorem. Moreover, $\bar{Z}_x$ is symmetric since *x* is an attainable allocation, and clearly $0 \in \bar{Z}_x$.

In Vind (1978) the concept of a simple market and a corresponding equilibrium notion are defined. A market is simple if it contains any finite sum of elements of itself. Clearly, the set $\tilde{Z}_x$ defined above is an example of a simple market. Analogously with Schmeidler and Vind's Theorem, Vind obtains an equivalence result. The paper by McLennan and Sonnenschein (1991) also contains an equivalence result based on the structure of the set of net trades available to the agents. McLennan and Sonnenschein define a strategic market game with a continuum of agents and give conditions under which all subgame perfect equilibria of the market game yield Walrasian allocations.

-
-
-

Bibliography

Anderson R. (1978). An elementary Core Equivalence Theorem. *Econometrica* **46**, 1483-1487. [Shows a very elegant and elementary Core decentralization result. See Section 5.1]

Anderson R. (1992). The Core in Perfectly Competitive Economies. *Handbook of Game Theory with Economic Applications*, Vol. 1 (eds. R. Aumann and S. Hart), 413-458. Amsterdam: North-Holland. [Contains an extensive survey of the Core convergence results. See Section 5.1]

Anderson, R. (1998). Convergence of the Aumann-Davis-Maschler and Geanakoplos Bargaining Sets. *Economic Theory* **11**, 1-37. [Shows that elements in the Aumann-Davis-Maschler Bargaining Set and in the Geanakoplos Bargaining Set can be approximately decentralized by prices in a large finite economy. See Section 5.2]

Anderson R., Trockel W., and Zhou L. (1997). Nonconvergence of the Mas-Colell and Zhou Bargaining Sets. *Econometrica* **65**, 1227-1239. [Gives an example that shows that elements in the Mas-Colell Bargaining Set cannot necessarily be approximately decentralized by prices in a large finite economy. See Section 5.2]

Anderson R. and Zame W. (1997). Edgeworth Conjecture with Infinitely Many Commodities: L1. *Econometrica* **65**, 225-273. [Shows approximately Core Decentralization with the commodity space L1. See Section 5.2]

Arrow K. and Debreu G. (1954). Existence of an Equilibrium for a Competitive Economy. *Econometrica* **22**, 265-290. [Gives the first general existence theorem for a Walrasian equilibrium in a finite economy. They convert the economy into a social system by adding an agent who sets prices. See Section 6]

Aumann R. (1964). Markets with a Continuum of Traders. *Econometrica* **32**, 39-50. [Introduces a continuum economy and proves the first Core equivalence theorem. See Section 4.1]

Aumann R. (1975). Values of Markets with a Continuum of Traders, *Econometrica* **43**, 611-644. [Introduces the set of Value allocations in an atomless economy and shows Value equivalence. See Section 4.3]

Aumann R. and M. Maschler (1964). *The Bargaining Set for Cooperative Games. In Advances in Game Theory*, (eds. M. Dresher, L. Shapley, and A. Tucker), 443-476. Princeton, N.J.: Princeton University Press. [Introduces the Bargaining Set of a cooperative game. See Section 5.2]

Aumann R.J. and Shapley L.S. (1968, 1974). *Values of Non-Atomic Games*, 333 pp. Princeton, N.J.: Princeton University Press. [In this book Aumann and Shapley develop a theory for Non-Atomic Games. See Section 4.3]

Bewley T. (1973). Edgeworth's Conjecture. Econometrica, 41, 425-454. [Obtains strong Core convergence results by imposing strong conditions on the support of the agents' characteristics. See Section 5.1]

Bewley T. (1973). The equality of the Core and the Set of Equilibria in Economies with Infinitely Many Commodities and a Continuum of Agents. *International Economic Review* **14**, 383-393. [Gives an extension of Aumann's Core Equivalence Theorem to the case where the commodity space is the set of essential bounded measurable functions on a measure space (L8). See Section 4.1]

Brown, D. and Robinson A. (1975). Nonstandard Exchange Economies. *Econometrica* **43**, 41-55. [Proves the Core equivalence theorem for a nonstandard exchange economy. See Section 7]

Debreu, G. (1959). *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, 114 pp. New York: John Wiley and Sons, Inc. [The classical textbook on general equilibrium theory. See Section 1]

Debreu, G. (1975). The Rate of Convergence of the Core of an Economy. *Journal of Mathematical Economics* **2**, 1-7. [Gives the rate of convergence for a sequence of replica economies. See Section 5.1]

Debreu, G. and Scarf H. (1963). A Limit Theorem on the Core of an Economy. *International Economic Review* **4**, 236-246. [States the first general Core convergence result. See Section 5.1]

Dierker, E. (1975). Gains and Losses at Core Allocations. *Journal of Mathematical Economics* **2**, 119-128. [Gives a price decentralization theorem for Core allocations. See Section 5.1]

Dubey P. and Neyman A. (1997). An Equivalence Principle for Perfectly Competitive Economies. *Journal of Economic Theory* **75**, 314-344. [Presents an axiomatic approach to the equivalence principle. See section 4.4]

Edgeworth, F.Y. (1881). *Mathematical Psychics-An Essay On The Application of Mathematics to The Moral Sciences*, 150 pp. London: Kegan Paul. [Shows that competition leads to Walrasian allocations. See Section 5.1]

Ellickson, B., Grodal B., Scotchmer S., and Zame W. (1999). Clubs and the Market. *Econometrica* **67**, 1185-1217. [Give a general Core equivalence result for club economies. See Section 4.1]

Ellickson B., Grodal B., Scotchmer S., and Zame W. (1999). *Clubs and the Market: Large Finite Economies*. Discussion Paper 99-05, University of Copenhagen. Forthcoming Journal of Economic Theory. [Extends Anderson (1978) to club economies. See Section 5.1]

Gabszewicz J. and Mertens J.-F. (1971). An Equivalence Theorem for the Core of an Economy whose Atoms Are Not "Too" Big. *Econometrica* **39**, 713-721. [Gives a Core equivalence theorem for economies with atoms using a condition on the relative size of the atoms. See Section 4.1]

Gabszewicz J. and Vial J.P. (1972). Oligopoly "`a la Cournot" in a General Equilibrium Analysis. *Journal of Economic Theory* **4**, 381-400. [Proves, under relative strong assumptions, that the Nash equilibria of economies where firms behave strategically converge to the Walrasian equilibria when the number of firms converges to +8. See Section 7]

Geanakoplos J. (1978). The Bargaining Set and Nonstandard Analysis. Chapter 3 of Ph.D. Dissertation. Department of Economics, Harvard University, Cambridge, MA. [Introduces the Geanakoplos Bargaining Set. See Section 4.2 and Section 5.2]

Grodal B. (1972). A Second Remark on the Core of an Atomless Economy. *Econometrica* 40, 581-583.

[Shows that coalitions that can improve can be restricted to consist of small groups of similar agents. See Section 4.1]

Grodal, B. (1975). The Rate of Convergence of the Core for a Purely Competitive Sequence of Economies. *Journal of Mathematical Economics* **2**, 171-186. [Gives the rate of convergence for a general sequence of economies. See Section 5.1]

Grodal B. (1988). *Bargaining Sets and Walrasian Allocations for Atomless Economies with Incomplete Preference*s. Discussion Paper 88-15, University of Copenhagen. [Contains equivalence results for the Mas-Colell Bargaining Set when consumers' have non-complete preferences. See Section 4.2]

Hildenbrand W. (1974). *Core and Equilibria of a Large Economy*, 251 pp. Princeton: Princeton University Press. [A classical textbook on Core convergence. See Section 5.1 and 7]

Kannai Y. (1966). Values of Games with a Continuum of Players. *Israel Journal of Mathematics* **4**, 54-58. [Defines the asymptotic Value of a game with a continuum of players. See Section 4.3]

Keiding H. (1974). *Cores and Equilibria in an Infinite Economy. In Computing Equilibria: How and Why*. (eds. J. ZLo´s and M. ZLo´s), 65-75. Amsterdam: North- Holland. [Studies Core equivalence in economies with an uncountable number of agents without imposing a measure space structure. Only finite coalitions are allowed to improve upon an allocation. See Section 7]

Mas-Colell A. (1989). An Equivalence Theorem for a Bargaining Set. *Journal of Mathematical Economics* **18**, 129-139. [Defines the Mas-Colell Bargaining Set for an atomless economy and proves an equivalence result. See Section 4.2 and Section 5.2]

McLennan A. and Sonnenschein H. (1991). Sequential Bargaining as a Non-cooperative Foundation for Walrasian Equilibria. *Econometrica* **59**, 1395-1424. [Defines a non-cooperative market game that leads to Walrasian outcomes. See Section 3.2]

Ostroy J. and Zame W. (1994). Nonatomic Economies and the Boundaries of Perfect Competition. *Econometrica* **62**, 593-633. [Shows that atomless economies do not necessarily give Core equivalence when the commodity spaces are infinitely dimensional. See Section 4.1]

Shapley L. (1953). A Value for n-Person Games. *Contributions to the Theory of Games II* (eds. H. Kuhn and A. Tucker), 303-307. Princeton, N.J.: Princeton University Press. [Defines the Shapley Value of a TU game. See section 4.3] Schjødt, U. and B. Sloth (1994). Bargaining Sets with Small Coalitions. International Journal of Game Theory 23, 49-55. [Studies Bargaining Sets when there are restrictions on the size of the coalitions that can object and counter object. See Section 4.2]

Schmeidler D. (1969). Competitive Equilibria in Markets with a Continuum of Trades and Incomplete Preferences. *Econometrica* **37**, 578-585. [Proves a Core equivalence theorem for atomless economies with non-complete preferences. See section 4.1]

Schmeidler D. (1972). A Remark on the Core of an Atomless Economy. *Econometrica* **40**, 579-580. [Shows that if an attainable allocation can be improved upon, then it can be improved upon with an arbitrarily small coalition. See Section 4.1]

Schmeidler D. (1980). Walrasian Analysis Via Strategic Outcome Functions. *Econometrica* **48**, 1585-93. [Defines a non-cooperative game in which the set of Nash equilibria equals the set of Walrasian allocations. See section 6]

Schmeidler D. and Vind K. (1972). Fair Net Trades. *Econometrica* **40**, 637-42. [Characterize the Walrasian allocations as attainable allocations having strongly fair net trades. See Section 3.2]

Shitovitz B. (1973). Oligopoly in Markets with a Continuum of Traders. *Econometrica* **41**, 467-501. [Investigates the connection between the Core and the Walrasian equilibria when the measure space of consumers has atoms. See Section 4.2]

Starr R. (1969). Quasi-Equilibria in Markets with Non-Convex Preferences. *Econometrica* **37**, 25-38. [Introduces the Shapley-Folkman Theorem. See Section 2.2]

Vind K. (1964). Edgeworth-Allocations in an Exchange Economy with Many Trades. *International Economic Review* **5**, 165-177. [Introduces an atomless economy in which coalitions are the basic actors and proves a Core equivalence theorem without assuming complete preferences. Moreover, introduces Lyapunov's Theorem into economics. See Section 2.1 and 7]

Vind K. (1972). A Third Remark on the Core of an Atomless Economy. *Econometrica* **40**, 585-586. [Shows that if an attainable allocation can be improved upon, then it can be improved upon with an arbitrarily large coalition. See Section 4.1]

Vind K. (1978). Equilibrium with Respect to a Simple Market. *Equilibrium and Disequilibrium in Economic Theory* (ed. G. Schwodiauer), 3-6. Dordrecht: D. Reidel Publishing Co. [Defines a simple market and proves an equivalence result. See Section 3.2]

Vind K. (1995). Perfect Competition or the Core. *European Economic Review* **39**, 1733-1745. [It is argued that Edgeworth did not define the Core but the set of exchange equilibria. See Section 5.1]

Biographical Sketch

Birgit Grodal is Professor of Economics at University of Copenhagen. She received Master's in Mathematics 1968 and gold medal for a dissertation in mathematical economics 1970 from the University of Copenhagen. She has done extensive research within economic theory and especially in general equilibrium theory. In particular she has contributed to the theory of the core, economies with imperfect competition, and economies with incomplete markets. Moreover, she has contributed to the theory of economies with clubs. She has a large number of publications, including papers in *Econometrica*, *Review of Economic Studies*, *Journal of mathematical Economics*, and *Economic Theory*. She has been associated editor of *Econometrica* and *Economic Theory*, and has been on the editorial board of *Journal of Mathematical economics*. She is fellow in the Econometric Society, and has been on the Council and the Executive Committee of the Econometric Society. She is member of Academia Europeae. She has been visiting researcher at several American and European Universities.