INVENTORY MODELS

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Contents

Summary

This article offers an introduction to the basic lines of research in inventory management: economic order quantity (EOQ) type models, dynamic economic lotsize models, periodic review stochastic demand models, and continuous review stochastic demand models.

1. Introduction

Inventory theory deals with the management of stock levels of goods with the aim of ensuring that demand for these goods is met. Most models are designed to address two fundamental decision issues: when a replenishment order should be placed, and what the order quantity should be. Their complexity depends heavily on the assumptions made about demand, the cost structure and physical characteristics of the system.

Inventory control problems in the real world usually involve multiple products. For example, spare parts systems require management of hundreds or thousands of different items. It is often possible, however, for single-product models to capture all essential elements of the problem, so it is not necessary to include the interaction of different items into the formulation explicitly. Furthermore, multiple-product models are often too unwieldy to be of much use when the number of products involved is very large. For this reason single-product models dominate the literature, and are used most frequently in practice. In the following, we therefore restrict attention largely to instances involving a single product.

Even when inventory models are restricted to a single product the number of possible models is enormous, due to the various assumptions made about the key variables: demand, costs, and the physical nature of the system. The demand for the product may be deterministic or stochastic; it may completely predictable, or predictable up to some probability distribution only; its probability distribution may even be unknown. Moreover, demand may be stationary or nonstationary, and may depend on economic factors that vary randomly over time.

The costs involved include ordering/production costs, which are either proportional to the order quantity or are more general. They may incorporate a setup cost, costs for holding the product in stock, and penalty costs for not being able to satisfy demand when it occurs. In addition, a service level approach may be used if it is too difficult to estimate penalty costs.

The stream of costs (or expected costs, if there is some uncertainty in demand and/or lead-times) over a finite or infinite horizon is minimized. The *average cost criterion* compares the order policies with regard to their average cost, while the *total cost criterion* compares order policies in relation to the present value of their cost-stream.

Inventory models are also distinguished by the assumptions made about various aspects of the timing and logistics of the model. Examples of these may include the following:

- The lead-time is often zero, but can also be of a fixed or random length.
- Back-ordering assumptions, which may be need to be made about the way that the system reacts when demand exceeds supply. The most common assumption is that all excess demand is back-ordered; the other extreme assumption is that all excess demand is lost. Mixtures of both the “backlogging” and the “lost sales” cases have been explored.
- Stock levels are reviewed continuously (over time) or periodically, maybe once a day or once a year, and are assumed to be known precisely or approximately.
- The quality of stored units, usually constant, is also allowed to change over time. Here we may distinguish between continuously deteriorating items and items with a fixed or random lifetime. Furthermore, the quality of incoming goods may be inconsistent due to the presence of random numbers of defective items.
- Different forms of ordering, such as emergency orders, as well as limited capacities of the resources used in production, are also considered.
- Inventory systems covering several locations, such as series systems, assembly systems, and distribution systems, differ in terms of their supply–demand relationships.

2. The Basic EOQ Model

We start with the classic *economic order quantity* (EOQ) model, which has formed the basis for a huge number of papers. In this simple model there is one product, which is replenished in continuous units. The demand is known with certainty and occurs at a constant rate λ; shortages are not allowed. The lead-time for each order is zero. The costs are stationary and consist of a fixed cost k, an ordering cost c per unit ordered, and a cost h per time unit that is charged for each unit of on-hand inventory.

On the basis of orders of a fixed size q, there is a cycle time (time between two successive arrivals of orders) of length $T = q/\lambda$. Since all cycles are identical, the average cost per time unit is then simply the total cost incurred in a single cycle divided by the cycle length, which is identical to

$$C(q) = \frac{k + cq + h\int_0^T (q - \lambda t)\, dt}{T} = \frac{k + cq + \frac{1}{2} hTq}{T} = \frac{k\lambda}{q} + c\lambda + \frac{1}{2}hq$$

and, as a function of q, becomes minimal for

$$q^* = \sqrt{2k\lambda / h}$$

a result known as the *economic order quantity*.

There are numerous variants and extensions of the basic EOQ model. Here we can only outline certain lines of research, and refer students to the Bibliography for more detailed information. Permitting back-orders enlarges the set of operating policies, and leads to a larger order quantity and a lower total cost compared with the EOQ model. If there is a deterministic lead-time, which is nonzero, then each order should be placed so that it is received exactly when the on-hand stock decreases to zero. The supply process in the EOQ model may result from a production process at constant rate $\mu > \lambda$. Then C(q) has the same form as in the EOQ model, with h replaced by $h(1-\lambda/\mu)$, and thus becomes smaller while q* becomes larger. In the basic EOQ model the variable cost c is constant for orders of all sizes. However, it is common for suppliers to offer price concessions for large orders. In fact, there are two kinds of discounts: incremental and all-units with cost functions $k + c(q)$, where $c(q) = c_0 q$ for $0<q<\chi$ and $c(q) = c_0\chi + c_1(q-\chi)$ for $q\geq\chi$, apply in the incremental case and $c(q) = c_0 q$ for $0<q<\chi$ and $c(q) = c_1 q$ for $q\geq\chi$ in the all-units case. A couple of EOQ-like calculations then suffice to find the optimal policy.

Learning effects are said to exist when the unit production cost or production time decreases as a result of experiences gained in the process. The implication of the lot-sizing problem is that the production cost is not constant over time. More general, extensions exist allowing cost parameters to change after a known future point in time, or to increase/decrease continuously at a fixed rate. A trend in demand may also be considered.

Up to now we have assumed that the goods received from the supply system are free from defects, and that the product remains in perfect condition throughout the process to final delivery to the customer. Now, let us first suppose that each batch that the supply system sends contains a fixed proportion of defective product. This case reduces to the EOQ model with appropriately adjusted parameters. Secondly, suppose that the supply system operates perfectly, but that defects arise while the product is held in inventory, as in the case of perishable goods such as food and medicines. The EOQ extensions primarily suppose exponential decay of the items; they use a differential equation to obtain the inventory on hand and, finally, a nonlinear program to obtain the approximate optimal order quantity.

The basic EOQ model assumes that the amount received is the same as the amount ordered. A production process is often imperfect, however, so the amount received (given lot size q) can be thought of as a random variable Y, for example, with first and second moment $E(Y|q)$ and $E(Y^2|q)$, respectively, resulting in $C(q) = c\lambda + [k\lambda + hE(Y^2|q)/2]/E(Y|q)$.

Most work of this kind has been extended to inventory systems involving several locations.

3. The Dynamic Economic Lotsize Model

We next consider a fixed-time inventory model with deterministic demand varying over time. Let $x_1, \dots, x_N$ be the known demand for the single product at the time points 1, …, N. We must meet all demand as it occurs: no back-orders or lost sales are permitted. We can order at each point, and we may carry inventory from one point to the next. Replenishment decisions take effect immediately: there is no lead-time. An order of size a_n at time n incurs a cost $c(a_n) = k_n + c_n a_n$, if $a_n > 0$, and $c(a_n) = 0$, otherwise. Further, there is a cost $h_n(s_n) = h_n \cdot s_n$ for holding inventory s_n at time n.

The objective is to schedule the order sizes $a_1, \dots, a_N$ so as to satisfy the demand $x_1, \dots, x_N$ at minimum total cost, which leads to the following nonlinear program

minimize $C(a_1, \dots, a_N) = \sum_{n=1}^{N} \left(c_n\left(a_n\right) + h_n\left(s_n\right) \right)$

subject to the initial condition $s_1 = 0$, the system dynamics

$s_{n+1} = s_n + a_n - x_n$ (n =1, …, N)

and the non-negativity restrictions

$s_2 \geq 0, \dots, s_{N+1} \geq 0, a_1 \geq 0, \dots, a_N \geq 0.$

The model possesses the remarkable property (the *zero-inventory property*) that an optimal solution $s_1^*, \dots, s_{N+1}^*$, $a_1^*, \dots, a_N^*$ exists such that $a_n^* > 0$ only exists when $s_n^* = 0$. Additionally, when the demands are all integer-valued, the only feasible values for a_n^* are 0, x_n, $x_n + x_{n+1}$, …, $x_n + \dots + x_N$, and the problem can then be reformulated and solved as a shortest-path problem.

Many extensions of the basic dynamic economic lotsize model have been developed. They embody lead-times, discounted costs, back-orders, and upper limits on order quantities.

4. Periodic Review Stochastic Demand Models

So far, we have supposed that the demand is completely predictable. In reality, however, there is often some uncertainty in demand. Uncertainty in demand degrades the performance of the inventory system; in particular, we are no longer able to foresee the ultimate effects of our actions, and thus may have to tolerate unplanned stockouts. This, however, will be true to some extent only when following an optimal order policy. We will now look at the structure of such an optimal policy, and the optimality criterion that it is based on. For this purpose, we adopt a discrete-time formulation as in Section 3. The system is observed at the beginning of a finite or infinite number N of periods (time points n = 1, 2, …, N); the demand X_1, X_2,…, X_N in the successive periods is independent, identically distributed with probabilities q(x), $x \in \{0, 1, \ldots\}$.

We consider cost c > 0 to purchase one unit of the product at the beginning of a period. Further, let h > 0 denote the cost per unit of the inventory remaining at the end of the period, and let p > c be a penalty cost per unit of unsatisfied demand. The lead-time is zero.

For N =1 we obtain the single period model, also known as the “newsboy” or “news-vendor” model. It applies directly to situations in which the (useful) life of a product is one period only, but forms the basis for nearly all multi-period models.

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Bibliography

Chikán A. (1990). *Inventories: Theories and Applications. A Selection of Papers Presented at the Fifth International Symposium on Inventories, Budapest.* 395 pp. Amsterdam: Elsevier. [Extensive bibliography.]

Graves S., Rinnoov Kan A.H.G., and Zipkin P.H. (eds.) (1993). *Logistics of Production and Inventory. Handbooks in OR and MS, Vol. 4.* 760 pp. Amsterdam: North-Holland. [Broad survey of the state of the art in logistics systems research.]

Guide V.D. and Scrivastava R. (1997). Repairable inventory theory: models and applications. *European Journal of Operational Research* **102**, pp. 1–20. [Recent review of repairable inventory theory.]

van Houtum G.J., Inderfuth K., and Zijm W.H.M. (1996) *Materials coordination in stochastic multi-echolon systems*. *European Journal of Operational Research* **95**, pp. 1–23. [Recent review of multi-echelon systems.]

Porteus E.L. (1990). Stochastic inventory theory. *Stochastic Models. Handbooks in OR and MS, Vol. 2* (eds. Heyman D.P and Sobel M.J.), pp. 605–652. [Recent survey on stochastic inventory control.]

Tayur S., Ganeshan R., and Magazine, M.J. (1999). *Quantitative Models for Supply Chain Management*. 885 pp. Norwell, MA: Kluwer. [Recent survey on supply-chain management.]

Waldmann, K.-H. (1984). Inventory control in randomly varying environments. *SIAM Journal on Applied Mathematics* **44**, pp. 657–666. [Early paper on inventory control in a random environment.]

Zipkin P.H. (2000) *Foundations of Inventory Management*. 514 pp. Boston, MA: McGraw-Hill. [Comprehensive summary, documenting the state of the art in book form]

Biographical Sketch

Karl-Heinz Waldmann is Professor of Mathematics and head of the Institute of Economic Theory and Operations Research at the University of Karlsruhe (Germany). He is working at the interface between applied probability, operations research, and computer science. His main areas of research are Markov decision processes and their applications, statistical quality control, and risk theory.