INTRODUCTION TO MATHEMATICAL ECONOMICS

T. Puu
CERUM, Umeå University, SE-90187 Umeå, Sweden

Keywords: attraction basins, business cycle theory, calculus of variations, capital value, central place theory, duopoly and oligopoly, dynamical processes, expected utility doctrine, general economic equilibrium, imperfect markets, internal rate of return, investment decisions, land use theory, linear and nonlinear programming, location theory, multiple coexistent attractors, optimization, migration, portfolio selection, rank-size relations, uniqueness and optimality.

Contents

Summary

Mathematical economics of today is too varied to be summarized in any meaningful sense. The reader will find this out in the following sections. If we want to define mathematical economics, it can only be done saying that it is the application of mathematical methods in economic theory. However, it is not clear which mathematical methods are useful and which are not, neither is there any unity of the different applications. Below, Schumpeter's warning that economists working with mathematics as a tool can not be defined as a school, as little as those who read Italian, will be cited.

For this reason, the historical origins of the use of mathematical analysis in economics, dating from early 19th Century, will be identified, as will some discussion about the applicability of mathematical methods to social sciences.

However, by mid 20th Century, the totality of applications of mathematics was still so limited in volume that treatises on mathematical economics could be published, despite the internal heterogeneity of the material. For this reason, the contents of such typical treatises will be discussed. As will become obvious, from that point, economists aiming at a unified science were more and more restricting the definition to general micro economics based equilibrium with a linear structure, sorting out mathematically modeled topics from the science, which did not fit in.

Some pains will be taken to discuss such areas of research, such as regional science, which were excluded from economics of the main stream.

1. Introduction

In the following chapters the reader will get a broad description of a great variety of the use of mathematical models in economics. In this introductory chapter therefore no attempt will be made at summarizing these, rather the focus will be on the nature of mathematical economics, its historical origins, and a cursory description of how mathematical economics was defined in the period 1940-1960, when it was firmly established in terms of what leading economists themselves identified as mathematical economics. Equally interesting as what was included in the definition is what was omitted in such authoritative accounts.

These omissions are quite interesting, and are the basis of some slight criticism below of the main stream economists' attitudes. As will be seen in following chapters, some of the omissions have later re-entered the literature, though much of such material was published only in various interdisciplinary journals, and never re-entered main stream economics.

Three different points of departure will be taken in this connection: (i) Joseph Schumpeter's account of mathematics and economic theory in his posthumous "*History of Economic Analysis*" 1952, (ii) the introduction and contents of RGD Allen's classical work "*Mathematical Economics*" of 1956, and (iii) a retrospective view by Gérard Debreu in his entry on "*Mathematical Economics*" in the "*New Palgrave Dictionary of Economics*" 1987.

It will also be considered to what extent the development in mathematical economics was in phase with contemporary development in mathematics, or just lagging behind. It will be noted that mostly economics just passively picked up mathematical methods of the past, even ignoring contemporary advances that would have been most useful, though there were a few exceptions, where economics could even be regarded as leading mathematics ahead, like physics always did.

2. The Origins of Mathematical Economics

Schumpeter takes a twin focus as origin for mathematical economics, dating from the early 19th Century, Johann Heinrich von Thünen's "*Der isolierte Staat in Beziehung auf Landwirtschaft und Nationalökonomie*" 1826, and Augustin Cournot's "*Recherches sur les principes mathématiques de la théorie des richesses*" of 1838.

2.1. von Thünen and Spatial Economics

It is interesting to note that among the sources mentioned, Schumpeter alone gives credit to von Thünen, the reason probably being that spatial economic theory remained a German tradition, and, for some reason, completely slipped out from the predominantly Anglo-Saxon main stream economics around 1960. It can be understood if for instance transportation was not included among the central issues of economics. However, such issues as trade, certainly remained central to economics, though von Thünen's explanation for specialization and hence for trade, building on the very general mathematical principle of transversality slipped out from economics. As a matter of fact, von Thünen's theory for specialization was much more general than was Ricardo's theory of comparative advantages, but to this day it has not been recognized as such. However, Schumpeter asserts that "*if we judge both men*" (von Thünen and Ricardo) "*by the ability of the purely theoretical kind then ... von Thünen should be placed above Ricardo or indeed above any economist of the period, with the possible exception of Cournot*".

Several attempts were later made to re-integrate spatial theory, first through Regional Science, with the foundation of the Regional Science Association in 1954 as an important landmark, though it remained a separate, interdisciplinary field, located somewhere in between economics and geography, giving rise to a sizeable number of newly founded interdisciplinary journals. Regional science also attracted many application oriented scientists with social interests from mathematics and physics, which economics did not, probably because it was more doctrinarian. A new attempt was recently tried in terms of the New Economic Geography. It still remains to be seen if this attempt is more successful in winning recognition from the general economists than Regional Science was.

2.2. Cournot and Imperfect Competition

Debreu too acknowledges Cournot as the founder of mathematical economics, but does not mention von Thünen. However, even the homage to Cournot is just passing. Debreu takes mathematical economics as more or less synonymous with the theory of general (competitive) economic equilibrium, where there is no place for imperfect competition.

Allen does not mention Cournot at all, but in his preface, the honest reservation is made that only Anglo-Saxon theoretical economics of his own time, with which he feels sufficiently familiar, will be covered. However, the fact that most of the later literature on imperfect competition actually was Anglo-Saxon, warrants an explanation of why market imperfections were discarded along with spatial issues from the agenda of mathematical economics.

2.3. Cournot, Walras, and Bertrand's Attack

Cournot's work, the foundation of duopoly and oligopoly theory, both in equilibrium terms and as a dynamical system, got a rejoinder later in the Century through Léon Walras's "*Eléments d'économie politique pure*" of 1874, which, however, focused general competitive equilibrium in a system of arbitrarily many interrelated markets, including some rudimentary dynamics (the well-known "*tâtonnemets*"). This fitted more than Cournot's market imperfections into the main frame of what economists increasingly regarded as the core of their discipline.

However, Cournot and Walras shared the conviction that a mathematical formulation of economic theories would be beneficial for clarity and conciseness of the science and both were attacked in a sweeping assault by Joseph Bertrand in "*Journal des savants*" ("*Théorie mathématique de la richesse sociale*") in 1883. Bertrand found some slight slips in Cournot's, reasoning, which rendered him the honor to get economists distinguish between Cournot oligopoly and Bertrand oligopoly on more or less equal terms. According to Schumpeter, Bertrand's argument was so confused that, had not Edgeworth cleared the whole thing up, no attention would have been paid to Bertrand at all. The main purpose of Bertrand's attack, of much interest in the present context, was to deny the significance or even possibility of the emergent mathematical economics. According to Schumpeter, Bertrand's argument was "*eagerly seized upon, as an authoritative condemnation, by people who understood neither mathematics nor economic theory*", and attracted much more interest than it deserved.

2.4. The Nature of Mathematical Economics

As to the nature of mathematical economics, we should note that economics is unique among the social sciences to deal more or less exclusively with metric concepts. Prices, supply and demand quantities, incomes, employment rates, interest rates, whatever studied in economics, are naturally quantitative metric concepts, where other social sciences need contrived concepts in order to apply any quantitative analysis. So, if one believes in systematic relations between metric concepts in economic theory, mathematics is a natural language in which to express them.

However, mathematics as a language is a slightly deceptive parable, as Allen points out in his preface. If it were merely a language, such as English, a mathematical text should be possible to translate into verbal English, just as an English text can be translated into, for instance, Russian. This is, however, not possible when mathematics enters the reasoning as an essential part of the argument.

Schumpeter too is keen to point out that mere representations of facts by figures, as in Francois Quesnay's "*tableau économique*" or Karl Marx's "*reproduction schemes*", is not enough for establishing a mathematical economics. Only when some tool, such as calculus, enters the reasoning in an essential way, can one speak of mathematical economics. Naturally, entering such technical methods can create an information gap, and occasionally cause misunderstanding and even disagreement between those who are and those who are not conversant with these.

Schumpeter's qualification held true in the case of von Thünen and Cournot, and later in the 19th Century, Walras.

2.5. The End of the 19th Century

Despite Bertrand's attack, mathematics gained force in applications to economic theory. We mentioned Frances Edgeworth, and should cite his most important work "*Mathematical Psychics*" of 1881, further the figurehead of the marginalist school William Stanley Jevons and "*The Theory of Political Economy*" 1871, Walras's follower Wilfredo Pareto and "*Manuel d'économie politique pure*" 1874, and finally Wilhelm Launhardt, with "*Mathemaische Begründung der Volkswirtschaftslehre*" 1885.

This is in no way intended to be an exhaustive picture of mathematical economics by the end of the 19th Century, just to give an impression of how many economists were involved in using mathematics by that time, so that what started at the beginning of the Century was firmly established by its end.

It is also worthwhile noting the diversity of the economic theories in which mathematics was used. Unlike our days, mathematical methods did not yet penetrate into each and every economic theory, but the diversity was such that Schumpeter warned one to regard mathematical economics as some kind of school or sect. In his own words, "*mathematical economists form no school in any meaningful sense of the term, any more than do those economists who read Italian*". It is not surprising that attempts to define mathematical economics become so diverse.

3. Mid 20th Century

In the first half or the 20th Century, mathematical modeling was widely used in theoretical economics. However, despite its aforementioned diversity, the total volume of applications was such that in 1956 RGD Allen could still write his classical "*Mathematical Economics*" in one, though sizeable volume. As mentioned, in the preface the reservation is made that only Anglo-Saxon theoretical economics current in

Allen's time would be covered. Given the general devastation of continental Europe in the war, including the dispersal of German speaking scientists to Britain and the US, makes the geographical limitation less important, provided state of the art in the period when the book was written rather than a historical perspective was chosen.

Such an enterprise as Allen's would seem totally unimaginable today, since mathematical methods penetrated almost every corner of economic theory. In the preface Allen also refers to the still ongoing debate about the applicability of mathematical methods to economics.

3.1. Remaining Opposition to Mathematical Economics

The usual argument against was that human behavior could not be "captured in mathematical formulas". Hence, the psychology of the agents, sometimes calculating, sometimes even irrational, made them incomparable to the elementary particles of matter whose behavior could be modeled by deterministic processes. Any theory or model of economics, in order to be the least true to reality, would therefore have to be much more complicated than the physical theories in which mathematical economists always took their inspiration. One is led to the suspicion that it was a mistake of economics to try to become the physics of social sciences.

In the present author's opinion, such sweeping comparisons are mistaken, not so much in *over*estimating the complexity of social phenomena as in *under*estimating the complexity of physical phenomena. Theorizing or modeling always is a heroic simplification to the purpose of distilling a few features from the unmanageable complexity of reality, all of whose interrelations cannot be analyzed in any intelligible way. Through the simplifications we can imagine to understand something of real phenomena, and in this sense physics and economics are alike.

Just consider Newton's theory of gravity and the interaction of two bodies solved in 1687, which makes us think we understand what goes on in the planetary system. The little step from two to three bodies challenged many optimistic attempts over 200 years, but proved evasive. By 1800, Laplace's still held the belief that a "sufficiently powerful mind", knowing the locations and momenta of all particles of matter in the universe, would be able to predict its future for all times to come. Unexpectedly, Poincaré in 1888 showed that even a restricted variant of the three-body problem could not be solved in closed form, and that the future of the system was unpredictable. Likewise, acoustics is built on one dimensional waves, which makes us think we understand the working of musical instruments, though neither strings nor air columns are one dimensional, and any closer approach to reality just complicates things beyond measure. The point is that in terms of complexity, physics is not so much behind economics.

3.2. Topics in Allen's Mathematical Economics

So, what topics does Allen's 700 page treatise cover? The book is split, in roughly equal parts, between dynamical macroeconomic models and microeconomic optimization and equilibrium models. Considerable sections are devoted to explaining the relevant mathematics: (i) Complex numbers and the closed form solution of linear second-order ordinary difference and differential equations as background for the business cycle models of multiplier-accelerator type. (ii) Matrix algebra for optimization by consumers and producers, and equilibrium for multi-market systems.

3.2.1. Macro-Dynamics

No doubt this stuff is Anglo-Saxon by origin. As for business cycle theory, PA Samuelson's seminal article "*The interaction of multiplier analysis and the principle of acceleration*" of 1939, its predecessors in various publications by Alvin Hansen, and its culmination in John Hicks's "*The Theory of the Trade Cycle*" of 1950, definitely belong to this category. This also holds true for the obvious Keynesian background, with Hicks's "*Mr. Keynes and the classics*" 1937 as an important intermediate link, as for other very similar models.

3.2.2. Multi-Market Equilibrium

The same is true for the second part of Allen's monograph. Again, Hicks and Samuelson are the figureheads, this time Samuelson through his "*Foundations of Economic Analysis*" 1947, completing Hicks's "*Value and Capital*" of 1939. It is interesting here to note a difference of British and American style - Hicks relegates the mathematics to appendices and keeps the main text verbal, whereas Samuelson integrates the mathematics in the main exposition. As we know, Keynes did not want to use mathematics at all, though he was most knowledgeable in mathematical methods, which, as an illustration to Allen's doubts about mathematics being a language, left some ambiguity concerning the system Keynes was describing, so opening up for several interpretations.

In the case of microeconomic optimization and multi-market equilibrium, the background is definitely non-Anglo-Saxon. No doubt it goes back to Walras's "*Eléments d'économie politique pure*" 1874, with Gustav Cassel's "*Theoretische Sozialökonomie*" 1918 and Abraham Wald's "*Über einige Gleichungssysteme in der mathematischen Ökonomie*" 1936 as important stepping stones.

However, in the one and a half decades 1935-1950, there is no doubt that Hicks and Samuelson had worked everything out in much more mathematical detail, adding also perspectives of comparative statics and dynamics to the equilibrium analysis.

3.2.3. Input Output Analysis and Linear programming

Allen's exposition is also strongly influenced by the newly emergent input-output analysis (Leontief, 1942), in a way a mathematization of Quesnay and Marx, and linear programming (Danzig, 1949). This gives a strong linear background to Allen's account of microeconomics, though also the classical smooth optimization models are touched upon. The detail given on matrix algebra is then further utilized to give an account of game theory, as founded by von Neumann and Morgenstern in 1944.

3.2.4. Linearity in the Analysis of Macro-Dynamics

The linear structure also applies to Allen's exposition of dynamics. It is linear differential and difference equations of the first and second order that are in focus. Though an "engineering approach" is taken in the discussion of some nonlinear business cycle models by Goodwin, Kalecki, and Phillips, it is mysterious that there is no trace of the perturbation and averaging methods, summarized as early as 1950 in Stoker's "*Nonlinear Vibrations in Mechanical and Electrical Systems*", which would have applied like hand in glove for the analysis of a model such as Hicks's with "floor" and "ceiling". See also Krylov and Bogoljubov "*Introduction to Nonlinear Mechanics*" 1949.

As a mathematician, though employed in a school of economics, Allen might have been likely to have known this stuff, which dates back to Rayleigh (1896), Duffing (1918), and van der Pol (1926), and, as stated, was fully developed by the time Hicks published his nonlinear floor/ceiling model. It is truly mysterious that these methods, adapted to deal with nonlinear oscillating systems in the pre-computer age, and by no means as abstract as the geometrical methods emerging from the Russian school, were never employed by economists. Allen's analysis of the Hicksian floor/ceiling model remains remarkably pedestrian.

3.3. Market Imperfections

Another curious feature in Allen's exposition of contemporary mathematical economics is the omission market imperfections. There existed an ample Anglo-Saxon literature, in a definitely mathematical dress which should have been known to Allen.

Cournot's work of 1838 was, as we saw, attacked by Betrand in 1883. The point of attack was that Cournot left some ambiguity as to the use of price as a means of competition. If price was used, then the possibility of cutting out the competitor through a marginal decrease of supply price would lead to essential instability of the model, at least provided the commodity was taken as homogenous. Therefore, more recent literature, departing from Cournot, assumed supply quantities to be the variables used as means of competition, and left the formation of market price given the supplies of all competitors an open issue.

3.3.1. The Hotelling Model

Two important solutions to this issue were proposed. The most intriguing was probably Harold Hotelling's "*Stability in competition*" 1929, where the commodity was taken as homogenous in view of consumers' preferences, but was put in a spatial setting with the competitors located at some distance from each other (consumers being continuously distributed over space), so that transportation costs provided them with local monopoly areas where competition took place at the boundaries. When relocation of the competitors was included in the model, then an essential instability arose, leading to the competitors eventually locating in the same point, so returning to the original Cournot model with its possible inherent undercutting instability.

In Lerner and Singer's rejoinder "*Some notes on duopoly and spatial competition*" 1937, the authors clearly realize that the problem was due to demand being fixed in quantity for each consumer, and totally inelastic to price in Hotelling's original model. As the consumers were supposed to always choose the cheapest commodity (mill price plus transportation charge) the insensitivity of the quantity demanded with respect to price was a little inconsequent. Lerner and Singer produced an impressive analysis of the Hotelling model when such sensitivity was introduced, if just in terms of a maximum reservation price. This was a way of removing some of the instability.

Further, Smithies in two papers "*Monopolistic price policy in a spatial market*" and "*Optimum location in spatial competition*" both 1941, argued for the use of a normal (linear) decreasing demand function. It is curious that Smithies claimed the integrations needed to obtain total demand as functions of mill price in such cases to be prohibitively complex. As a matter of fact, Hotelling's space is one dimensional (an intercontinental railway as he suggests), so the closed form integrals are quite easy to obtain. Smithies's claim is a testimony of the lack of integration techniques in the average economist's equipment, which is curious compared to the high sophistication in differentiation techniques.

3.3.2. Heterogeneous Commodities

An alternative to assuming a spatial setting with transportation costs, one could just assume the commodities supplied to be competitive, though perceived as slightly different by the consumers. This idea was fully developed by Edward Chamberlin in "*The economics of monopolistic competition*" 1933. There, is, however, not so much intriguing mathematical analysis pertaining to this solution, so it is sufficient to just mention it.

3.3.3. Imperfect Competition

Contemporaneous was Joan Robinson's "*The Economics of Imperfect Competition*" 1933, where important aspects of monopoly, including price discrimination, were elaborated. The most intriguing issue from a retrospective viewpoint was the analysis of cases with multiple optima for the monopolist. Robinson assumed a kink in the linear demand function, due to the fact that when the price charged for a commodity was high, then it tended to become a luxury with high elasticity of demand. With lowered prices, new groups of consumers could afford the commodity, and the elasticity would increase considerably.

As a consequence, the marginal revenue curve would have a discontinuity, and jump up, with several intersections with the marginal cost curve. These intersections provided local profit maxima, among which one would be the global. This is one of the few instances of global issues in mid 20th Century economics. Robinson assumes the monopolist to be myopic, and stick to any local optimum that happened to be chosen, without looking for any other globally better solution. Such optimum search might be introduced through some dynamical process, but Robinson abstained from doing this.

3.3.4. Multiple Equilibria in Duopoly

A dynamical aspect was, however, added by Tord Palander in "*Instability in competition between two sellers*" 1936, elaborated in more detail in a Swedish article

1939, where he used the Robinson kinked demand model in duopoly. As the discontinuous marginal revenue curves shifted due to the quantity the competitor supplied, the global profit maximum of the duopolist could jump between the local maxima, so, as a rule, the reaction functions came in disjoint pieces.

Note that this is not the case baptized kinked-demand oligopoly, in which the kink is the other way, and due, not to different demand elasticities of different consumer groups, but to the assumption that competitors react to price cuts, but not to price raises. This is assumed to lead to extreme price stability, but the mathematics involved is rather trivial.

Such reaction functions that Palander discussed could intersect in two points or none at all. In the case of two intersections, Palander also detected the possibility of a two-period oscillation as a third alternative. This probably was the first mathematical economic model with multiple equilibria. Palander was surprisingly accurate for the time in describing the basins of the two fixed point attractors, and the third oscillating attractor. He also considered the case where there was no intersection between the reaction functions, and hence no Cournot equilibrium at all. For this case he characterized a three-period oscillation.

It is interesting to note that the same year as Palander, 1936, Wald in his aforementioned article, also constructed oligopoly models with multiple attractors - more precisely a non-denumerable infinity of them. It is curious that, though Wald's article is mentioned as an important stepping stone to proving the existence of general economic equilibria in Walrasian systems, the second half of his voluminous paper, devoted to issues of imperfect competition, seems no longer to be read.

3.3.5. Conjectural Variations

It is also necessary to mention Ragnar Frisch "*Monopole - polypole, la notion de force dans l'économie*" 1933, where he elaborated the role of expectations in oligopoly theory in terms of "conjectural variations", expectations of the reaction functions by the competitors which did not need to agree with the true ones.

Another important contribution to the expectational side of oligopoly was Heinrich von Stackelberg's contribution "*Probleme de unvollkommenen Konkurrenz*" 1938, where he introduced the concepts of price leader and price follower, the price leading duopolist learning the reaction function of the competitor, and taking it in explicit account in calculating his optimum. Of course, either duopolist could take this choice, and it worked if the other actually followed the reaction function. If both followed their reaction functions, then the original Cournot solution was retrieved. If, however, both decided to become leaders, then the situation was just unstable, like in Hotelling's price cutting, and the result was economic warfare.

-
-
-

Bibliography

Allen R.G.D. (1956). *Mathematical Economics*. Macmillan, London [The most comprehensive state of the art advanced textbook on topics in mathematical economics in mid 20th Century].

Allen R.G.D. (1938). *Mathematics for Economists*. Macmillan, London [An early textbook on mathematics for economics students].

Arrow K.J. and Debreu G. (1954). Existence of an equilibrium for a competitive economy. *Econometrica* **22**:265-290 [Final proof of existence of general economic equilibrium, in terms of equilibrium prices and clearing of all markets, in a system of an arbitrary number of interdependent markets].

Arthur B. (1990). Positive feedbacks in the economy. *Scientific American* February 92-99 [Summary criticism of economists' notion of uniqueness and optimality of equilibria. Arthur advocates cases with several coexistent attractors].

Beckmann M.J. (1952). A continuous model of transportation. *Econometrica* **20**:643-660 [The most concise and elegant model for equilibrium in a spatially extended market and the corresponding flow of commodities].

Berck P. and Sydsaeter K. (1991). *Economists' Mathematical Manual*. Springer, Berlin [A short-list of formulas and theorems useful for economists and results from mathematical economics].

Bertrand J. (1883). Théorie mathématique de la richesse sociale. *Journal des Savants* **48**:499-508 [Fierce attack on Cournot and Walras who advocated mathematical economic modeling].

Boulding K.E. (1934). The theory of a single investment. *The Quarterly Journal of Economics* **49**:475-81 [Insightful discussion of the nature of the internal rate of return on an investment and its relation to the capital value criterion].

Chamberlin E.H. (1933) *The Economics of Monopolistic Competition*. Harvard University Press, Cambridge, Mass. [Final statement of the theory of heterogeneous commodities and monopolistic competition].

Carlson S. (1939). *A Study on the Pure Theory of Production* . King, London [An early study of the theory of production, like Schneider's inspired by Frisch].

Cassel G. (1918) *Theoretische Sozialökonomie* Winter, Leipzig [Comprehensive development of Walras's theory of general equilibrium].

Courant R. (1936-7) *Differential and Integral Calculus*. Wiley, New York [Leading textbook on calculus in mid 20th Century].

Cournot A. (1838) *Recherches sur les Principes Mathématiques de la Théorie des Richesses*. Hachette, Paris [One of the first books on mathematical economics, dealing with the theory of duopoly and oligopoly].

Davis P.J. and Hersh R. (1983). *The Mathematical Experience*. Penguin, London [A fascinating book on the history of mathematics].

Debreu G. (1959). *Theory of Value*. Wiley, New York [Elaboration in great detail of such linear structure in production and preferences that the existence of a unique general economic equilibrium can be proved through various fixed point theorems].

Debreu G. (1987). Mathematical Economics. *The New Palgrave Dictionary of Economics*, Vol. 3 (ed. by J. Eatwell, M. Milgate, and P. Newman). Macmillan, London [Debreu's entry in the most authoritative encyclopaedia on economics and economists].

Devaney R. (1989). Film and video as a tool in mathematical research. *The Mathematical Intelligencer* **11**:33- [A leading chaos theorist's account of using motion pictures composed of computer generated sequences of graphs to understand bifurcations of attractors and basins due to parameter changes].

Edgeworth F.Y. (1881), *Mathematical Psychics*. (reprinted 1967), Kelley, New York [One of the most comprehensive early mathematical economics texts, containing, among other things, a clarification of competition through prices in Cournot's duopoly model].

Fisher I. (1930). *The Theory of Investment, As Determined by Impatience to Spend Income and Opportunity to Invest It*. New York [First clear attempt to base the conventional capital value calculations used in investment decisions on a firm theoretical basis through relating them to multi-period consumption theory].

Frisch R. (1926-). Innledning til Produksjonsteorien. *Memoranda fra Sosialoekonomisk Institutt*, Oslo [Various editions developing the theory of production. Published only 1962 in Norwegian, 1963 by Dunod, Paris in French translation and in English only 1965 as *Theory of Production* by Reidel, Dordrecht. Meanwhile inspired many treatises, such as Schneider 1934 and Carlson 1939].

Frisch R. (1933). Monopole - polypole, la notion de force dans l'économie. *Nationaloekonomisk Tidskrift* **71** tillaeggshaefte [Development of expectations in Cournot's oligopoly model through learning the behavior of the consumers and the competitors].

Frisch R. (1934) Statistical confluence analysis by means of complete regression systems. *Memoranda fra Sosialoekonomisk Institutt*, Oslo [Criticism of using statistics to merely correlate actual co-variations of economic variables].

Gandolfo G. (1997) *Economic Dynamics*. (3rd Edition). Springer-Verlag, Berlin [Leading textbook on economic dynamics, focused on linear systems in earlier editions, though augmented with some basics of nonlinear models].

Goodwin R.M. (1949). The business cycle as a self-sustaining oscillation. *Econometrica* **17**:174-185 [Business cycle model in continuous time, generalizing Harrod's growth model to a second order process. Makes the investment function nonlinear, and obtains a limit cycle].

Goodwin R.M. (1951). The nonlinear accelerator and the persistence of business cycles. *Econometrica* **19**:1-17 [Further development of last item].

Harrod R.F. (1948) *Towards a Dynamic Economics*. Macmillan, London [First statement of economic growth theory, using a linear savings function, and a fixed proportion technology whereby the stock of capital becomes proportional to real income. Obtains a "balanced growth" through equating savings to investments, but realizes that the equilibrium trajectory is unstable].

Hicks J. (1937). Mr. Keynes and the 'classics'. *Econometrica* **5**:147-159 [Most influential mathematical interpretation of the Keynesian macroeconomic model].

Hicks J. (1939) *Value and Capital*. Clarendon Press, Oxford [Contemporary mathematical study of most issues in microeconomics].

Hicks J. (1950). *The Theory of the Trade Cycle*. Oxford University Press, Oxford [Develops the multiplier-accelerator model by Samuelson to a nonlinear format through using "floor" (lower limit to disinvestment when no capital is replaced) and "ceiling" (upper limit when other inputs than capital become scarce].

Hirshleifer J. (1958). On the theory of the optimal investment decision. *The Journal of Political Economy* **66**:329- [Rejoinder to Fisher's investment theory, developing the claim that capital value criteria only make sense in a perfect capital market with just one rate of interest for all transactions].

Hotelling H. (1921). A Mathematical Theory of Migration. MA Thesis, University of Washington, republished in *Environment and Planning* **A10**:1223-1239 [A model combining logistic growth with spatial diffusion from densely to sparsely populated areas].

Hotelling H. (1929). Stability in competition. *Economic Journal* **39**:41-57 [Puts Cournot's duopoly model with a homogenous commodity in a spatial context, whereby the competitors become monopolists in their proper market areas but compete at the market boundary. If relocation is possible, the Hotelling model implies that the competitors place themselves in the midpoint of the fixed one-dimensional location space interval].

Jevons W.S. (1871). *The Theory of Political Economy*. London [Leading treatise of the "marginalist" school, using basic calculus for optimization].

Keynes J.M. (1936). *The General Theory of Employment, Interest, and Money*. Macmillan, London [A completely new theory of the economy, formulated in terms of economic aggregates, such as national income, investment, consumption, employment, etc. Negates the influence of monetary issues, and explains sustained involuntary unemployment. Though not mathematically formulated, the verbal description could be translated to precise mathematical models, for instance by Hicks].

Knight F. (1921) *Risk, Uncertainty, and Profit*. Houghton, Boston [Early discussion of uncertainty in investment theory].

Krylov N. and Bogoljubov N. (1949). *Introduction to Nonlinear Mechanics*. Princeton University Press, New York [The basic text on averaging methods for nonlinear differential equations].

Launhardt W. (1885). *Mathematische Begründung der Volkswirtschaftslehre*. W. Engelmann, Leipzig [Early comprehensive statement of mathematical economics. Formulates the location problem for a firm with pre-located input sources and otput market, and discusses the market boundaries for several firms supplying close substitute goods].

Lerner A.P. and Singer H.W. (1937). Some notes on duopoly and spatial competition. *Journal of Political Economy* **45**:145-186 [A very detailed and innovative discussion of the Hotelling duopoly model when some elasticity of demand is introduced in terms of a reservation price].

Lutz F.A. (1945). The criterion of maximum profits in the theory of investment. *The Quarterly Journal of Economics* **60**:56- [Discussion of the capital value criterion for investment decisions].

Lutz F. and Lutz V. (1951). *The Theory of Investment of the Firm*. Princeton [Studies the relations between internal rates of interest for bonds of different durabilities, based on an assumption of equivalence for one long and a sequence of short investments].

Lösch A. (1940). *Die räumliche Ordnung der Wirtschaft*, Fischer, Jena [A penetrating study of location lattices, market area shapes, and nesting of several layers of market areas of different mesh sizes].

Markowitz H. (1959) *Portfolio Selection*. Wiley, New York [Uses the expected utility doctrine with a quadratic utility function to derive optimal mixes of share portfolios on the basis of mean and variance in expected future statistical share price distributions].

Moses L. (1958). Location and the theory of production. *Quarterly Journal of Economics* **72**:259-272. [Solution to the Launhardt-Weber location problem when substitutability of inputs is assumed].

Okubo, A. (1980). *Diffusion and Ecological Problems*, Springer-Verlag [Discusses ecological diffusion models, originated by Skellam's model, identical with Hotelling's migration model, though re-invented 30 years later].

Palander T.F. (1936). Instability in competition between two sellers. Abstracts of papers presented at the research conference on economics and statistics held by the Cowles Commission at Colorado College. *Colorado College Publications* General Series 208, Studies Series 21 [Discusses oligopoly with disjoint marginal revenue functions (kinked demand), as suggested by Joan Robinson. The reaction functions come in disjoint pieces and can have several or no intersections].

Pareto V. (1874). *Manuel d'économie politique pure*". Lausanne [Development of Walras's general equilibrium model. Introduces ordinal utility, and reduces interpersonal comparabilities to cases where at least one is better off without anybody being worse off (Pareto optimality)].

Phillips A.W. (1954). Stabilisation policy in a closed economy. *The Economic Journal* **64**:290-323 [Business cycle model of second order formulated in continuous time].

van de Pol P. (1926). On relaxation oscillations. *Philosophical Magazine* **2**:978-992 [Model of oscillations in electrical circuits, very similar to Hicks's floor/ceiling model, with regimes alternating between storage of energy and sudden "relaxation"].

Ramsey F.P. (1928). A mathematical model of saving. *Economic Journal* **38**:543-559 [One of the very few economics applications of the calculus of variations, designed for the planning of saving over a prolonged period in continuous time].

Robinson J. (1933). *The Economics of Imperfect Competition*. Macmillan, London [Penetrating study of monopoly and pricing policies, such as price discrimination. Introduces demand functions where at lowered price new groups of consumers can afford the commodity and the elasticity increases. This produces a jumping marginal revenue curve and several intersections with the marginal cost curve, i.e., multiple profit maxima].

Rosser J.B. (2000) *From Catastrophe to Chaos: A General Theory of Economic Discontinuities*. Kluwer, Boston [Only existent comprehensive survey of nonlinear models in economics].

Samuelson P.A. (1939). The interaction of multiplier analysis and the principle of acceleration. *Review of Economics and Statistics* **21**:75-78 [First formulation of the interaction of a linear consumption function with an investment function where investments are proportional to the change of income. Produces damped or explosive oscillations to explain business cycles].

Samuelson P.A. (1947). *Foundations of Economic Analysis*. Harvard University Press, Harvard, Mass. [Sometimes regarded as the beginning of modern mathematical economics. Solves issues left open in Hicks's "Value and Capital", and discusses a large selection of dynamical problems].

Schneider E. (1934). *Theorie der Produktion*. Springer, Wien [Early treatise on production theory inspired by Frisch].

Schneider E. (1944). *Investering og Rente*. Copenhagen. German edition (1964): *Wirtschaftlichkeitsrechnung, Theorie der Investition*. Tübingen [A comprehensive statement of the theory of investment, though neglecting Fisher's suggestion to base it on multi-period consumption theory].

Schumpeter J. (1952). *History of Economic Analysis*. Oxford University Press, Oxford [The standard reference on the history of economics from the medieval doctors to Schumpeter's own time].

Slutsky E. (1915). Sulla teoria del bilancio del consumatore. *Giornale degli Economisti* **51**:1-26 [First comprehensive study of the comparative statics for the consumer's optimum].

Smithies A. (1941). Optimum location in spatial competition. *Journal of Political Economy* **49**:423-439 [Proposition to deal with the Hotelling duopoly instability problem using elastic demand in terms of a linear demand function].

Smithies A. (1941). Monopolistic price policy in a spatial market. *Econometrica* **9**:63-73 [Continuation of the previous item, including a claim that the integration to obtain total demand for each competitor as a function of the mill prices of itself the competitor is too complex to be dealt with analytically].

von Stackelberg H. (1938). Probleme der unvollkommenen Konkurrenz. *Weltwirtschafliches Archiv* **48**:95-114 [Statement of the problem of "leadership" in duopoly. When one firm takes the Cournot reaction function of the competitor in explicit account and calculates its own best action it is attempting to become "leader". Two such stable "leader/follower" pairs can thus be formed. If both competitors try to become leaders, the system is not consistent as both are mistaken about the other's behavior].

von Stackelberg H. (1938). Angebot und Nachfrage in der Produktionswirtschaft. *Archiv für mathematische Wirtschafts- und Sozialforschung* **4**:73-99 [A thorough study of production theory, comparable to Slutsky's of 1915].

Stoker J. J. (1950). *Nonlinear Vibrations in Mechanical and Electrical Systems.* Wiley, New York [Basic source material on perturbation methods for nonlinear dynamical systems].

Strutt J.W., Lord Rayleigh. (1877). *Theory of Sound*. London [Classical study of the nature of sound waves. The model for the vibration of the bowed string in terms of alternating regimes of forced and free oscillations is a dynamical system which resembles the nonlinear business cycle theories set up by economists].

von Thünen J.H. (1826). *Der isolierte Staat in Beziehung auf Landwirtschaft und Nationalökonomie.* Hamburg [One of the first books on mathematical economics. Most known for its statement of specialization and trade in a rural landscape around a market place.]

Wald A. (1936) Über einige Gleichungssysteme in der mathematischen Ökonomie. *Zeitschrift für Nationalökonomie* Dezember 637-670 [Statement of the mathematical conditions under which the existence of general economic equilibrium could be proved. Later elaborated by Arrow and Debreu. The second part of the article is devoted to imperfect markets, i.e., oligopoly, and conditions under which multiple equilibria could coexist].

Walras L. (1874-7) *Eléments d'économie politique pure*. Paris [First formulation of the problem of "general equilibrium", i.e., the simultaneous clearance of all excess supplies and demands through a set of equilibrium prices in an arbitrary number of interdependent markets].

Weber A. (1909) *Über den Standort der Industrie*. Tübingen [The classical statement of location theory, in terms of the "Weberian location triangle", though the problem and its solution were stated a few decades earlier by Launhardt].

Biographical Sketch

Tönu Puu was born in Tallinn (Estonia) 1936. He studied at Uppsala University (Sweden) and finished his PhD in economics 1964.

He was appointed to the Chair of Economics at Umeå University by Royal patent 1971. After emeritation 2001, he works as research professor at the Centre for Regional Science at Umeå University. He published 15 monographs, including *Attractors, Bifurcations, and Chaos*, (Berlin: Springer, 2000, 2003), *Mathematical Location and Land Use Theory* (Berlin: Springer, 1997, 2003), and *Arts, Sciences, and Economics* (Berlin: Springer, 2006) and over 100 articles.

Prof. Puu was (or is) member of the editorial board of several scientific journals, including: Annals of Regional Science; Regional Science and Urban Economics; Journal of Regional Science; Networks and Spatial Theory; Chaos, Solitons & Fractals; Discrete Dynamics in Nature and Society; and International Journal of Shape Modeling. He was also founder and director of the Nordic Baroque Music Festival 1987-2001.