NONLINEAR OUTPUT REGULATION

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Summary

The problem of nonlinear output regulation is examined. It is shown that the solvability of the problem amounts to the solvability of a set of partial differential equations. The notion of immersion and the internal model principle are presented.

1. The Problem of Output Regulation

One of the most relevant problems in control theory is to force via feedback a system to achieve a prescribed steady-state response under the action of every external command in a given family. In this typology fit problems such as that of obtaining a controlled output $y(t)$ to asymptotically track a reference function of time $y_{ref}(t)$ belonging to an assigned class of signals as well as the problem of asymptotically rejecting the effect on the output $y(t)$ of a disturbance $w(t)$ ranging over a certain family of disturbances. In both cases the problem boils down to have the *tracking error*

$$e(t) = y(t) - y_{ref}(t),$$

that is the discrepancy between the controlled output and the reference signal, decay to zero as time goes by for every reference output and any disturbance in some given family of functions.

The problem in question can be formalized in the following way. Consider the dynamics of a *process* described by nonlinear equations

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$$\dot{x} = f(x, w, u), \tag{1}$$

where $x(t)$ is the *state* of the process ranging over a neighborhood U of the origin in $\mathbb{R}^n$, $u(t)$ is a *control* input taking on values in $\mathbb{R}^m$ and $w(t) \in \mathbb{R}^r$ is an *exogenous* vector-valued signal which includes references to be tracked and/or disturbances to be rejected. In addition to the process dynamics we consider the equation

$$e = h(x, w) \tag{2}$$

expressing the error variable $e(t) \in \mathbb{R}^m$. Functions $f(x, w, u)$ and $h(x, w)$ are assumed to be *smooth* and to satisfy the conditions $f(0,0,0) = 0$ and $h(0,0) = 0$.

The family of exogenous signals $w(t)$ that act on the process, is obtained by considering all the solutions of a (possibly nonlinear) homogeneous differential equation

$$\dot{w} = s(w) \tag{3}$$

as the initial condition $w(0)$ ranges over a neighborhood W of the origin in $\mathbb{R}^r$. This system, which is interpreted as a generator of all possible exogenous inputs, is called an *exosystem*.

It is assumed that function $s(w)$ is smooth and such that $s(0) = 0$, and hence, recalling an analogous hypothesis introduced earlier on the functions defining the plant and the error equation, the combined system (1), (3) with $u = 0$ has an equilibrium state $(x, w) = (0, 0)$, at which the error is zero.

The property of *output regulation* is the property of the controlled system for which the tracking error $e(t)$ converges to zero as t goes to infinity despite of the action of exogenous signals. This property is particularly meaningful, when the exogenous signals are “persistent" in time, that is, no exogenous signal decays to zero as time tends to infinity. This happens, for instance, for any periodic (and bounded) function of time. In these cases, in fact, the system may exhibit a “steady-state response" that is itself a persistent function of time, and whose characteristics depend entirely on the specific input imposed on the system and not on the initial conditions of the system. To ensure that the exogenous inputs generated by the exosystem (3) are bounded, it is enough to assume that the point $w = 0$ is a Lyapunov stable equilibrium of $s(w)$ and to choose the initial condition $w(0)$ sufficiently close to the origin, namely in some suitable neighborhood $W^0 \subset W$ of the origin. In order to guarantee the “persistency property" of the exogenous signals, it is useful to assume that every point w in W^0 is *Poisson stable*. Indeed, by definition, this is equivalent to say that every exogenous signal which originates from w, i.e. every exogenous signal $w(t)$ which is a solution of $\dot{w} = s(w)$ and satisfies the initial condition $w(0) = w$, passes arbitrarily close to w for arbitrarily large values of time (in both forward and backward directions), and hence cannot decay to zero as time goes to infinity.

The two properties that the map $s(w)$ admits a Lyapunov stable equilibrium point at $w = 0$ and that there exists a neighborhood of Poisson stable points around $w = 0$ will be together referred as to the *neutral stability* property.

Remark. A necessary condition for the neutral stability property to hold is that the matrix defining the linear approximation of $s(w)$ at $w = 0$, i.e. matrix

$$S = \left(\frac{\partial s}{\partial w}\right)_{w=0}$$

has all its eigenvalues on the imaginary axis.

The control action to system (1) is to be provided by a *feedback controller*, which processes the information originated from the process to generate the appropriate control action. The structure of the feedback controller depends on the information pattern available for feedback. There are two possible situations. The most favorable one from the viewpoint of feedback design occurs when all the components of the state variable x and of the exogenous input w are available for measurements. In this case, the controller is said to have access to *full information*, and it can be constructed as a *memoryless* feedback, whose output u is a function of the states x and w of the process and, respectively, of the exosystem, that is

$$u = \alpha(x, w) \tag{4}$$

with $\alpha(x, w)$ smooth and satisfying $\alpha(0,0) = 0$.

A more realistic, and rather common, situation is the one in which the set of measured variables includes only the components of the tracking error e. If this is the case, the controller is said to have access to *error feedback*, and it is convenient to synthesize the control signal by means of a *dynamic* nonlinear system of the form

$$\begin{aligned} \dot{\xi} &= \eta(\xi, e) \\ u &= \theta(\xi), \end{aligned} \tag{5}$$

where ξ is the internal state defined in a neighborhood Ξ of the origin of $\mathbb{R}^{\nu}$, ν is an appropriate positive integer, and $\eta(\xi, e)$, $\theta(\xi)$ are smooth functions which are zero at $(\xi, e) = (0,0)$ and $\xi = 0$, respectively.

In the case of full information, the interconnection of system (1), (3) with controller (4) yields a closed-loop system described by the equations

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$$\begin{aligned} \dot{x} &= f(x, w, \alpha(x, w)) \\ \dot{w} &= s(w), \end{aligned} \tag{6}$$

for which the point $(x, w) = (0, 0)$ is an equilibrium, whereas in the case of error feedback, the interconnection of (1), (2), (3) with (5) yields the closed-loop system

$$\begin{aligned} \dot{x} &= f(x, w, \theta(\xi)) \\ \dot{\xi} &= \eta(\xi, h(x, w)) \\ \dot{w} &= s(w), \end{aligned} \tag{7}$$

for which the point $(x, w, \xi) = (0, 0, 0)$ is an equilibrium.

If the exosystem is neutrally stable and system

$$\dot{x} = f(x, 0, \alpha(x, 0)) \tag{8}$$

is *asymptotically stable in the first approximation*, then the response of the closed-loop system (6) converges – from any initial condition $(x(0), w(0))$ in a suitable neighborhood of the origin $(0, 0)$ – to a well-defined *steady-state response* as time goes to infinity, and this steady-state response depends only on $w(0)$ and not on $x(0)$. If, additionally, the steady-state response is such that the corresponding error tracking is identically zero, then the closed-loop system has the desired property of output regulation. This motivates the following definition.

Full Information Output Regulation Problem Given a process described by nonlinear equations of the form (1) and a neutrally stable exosystem of the form (3), find, if possible, a mapping $\alpha(x, w)$ for which:

(S_{FI}) The equilibrium $x = 0$ of (8) is asymptotically stable in the first approximation.

(R_{FI}) There exists a neighborhood $V \subset U \times W$ of $(x, w) = (0, 0)$ such that, for each initial condition $(x(0), w(0))$ in V, the solution of (6) satisfies

$$\lim_{t \to \infty} h(x(t), w(t)) = 0.$$

Analogously, if the exosystem is neutrally stable and system

$$\begin{aligned} \dot{x} &= f(x, 0, \theta(\xi)) \\ \dot{\xi} &= \eta(\xi, h(x, 0)) \end{aligned} \tag{9}$$

is *asymptotically stable in the first approximation*, then the response of the closed-loop system (7) converges – from any initial condition $(x(0), \xi(0), w(0))$ in a suitable neighborhood of the origin $(0, 0, 0)$ – to a well-defined *steady-state response* as time

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goes to infinity, and this steady-state response depends only on $w(0)$ and not on $(x(0),\xi(0))$. If, additionally, the steady-state response is such that the corresponding error tracking is identically zero, then the closed-loop system has the desired property of output regulation.

Error Feedback Output Regulation Problem Given a process described by nonlinear equations of the form (1) and a neutrally stable exosystem of the form (3), find, if possible, an integer ν and two mappings $\eta(\xi,e)$, $\theta(\xi)$ for which:

($\mathrm{S_{EF}}$) The equilibrium $(x,\xi)=(0,0)$ of (9) is asymptotically stable in the first approximation.

($\mathrm{R_{EF}}$) There exists a neighborhood $V\subset U\times\Xi\times W$ of $(x,\xi,w)=(0,0,0)$ such that, for each initial condition $(x(0),\xi(0),w(0))$ in V, the solution of (7) satisfies

$$\lim_{t\to\infty} h(x(t),w(t))=0\,.$$

The requirements of asymptotic stability in the first approximation listed in the formulation of the two output regulation problems clearly demand properties of *stabilizability* and *detectability* of the linear approximation at the origin of the closed-loop systems. In order to make this precise, one can rewrite the controlled system (6) as

$$\begin{aligned}\dot{x} &= (A+BK)x+(P+BL)w+\phi(x,w)\\ \dot{w} &= Sw+\psi(w)\,,\end{aligned}\qquad(10)$$

having denoted with $\phi(x,w)$ and $\psi(w)$ higher order terms which vanish at the origin along with their first-order derivatives, and with A,B,P,K,L,S the following matrices:

$$A=\left(\frac{\partial f}{\partial x}\right)_{(x,w,u)=(0,0,0)}\qquad B=\left(\frac{\partial f}{\partial u}\right)_{(x,w,u)=(0,0,0)}\qquad P=\left(\frac{\partial f}{\partial w}\right)_{(x,w,u)=(0,0,0)}$$
$$K=\left(\frac{\partial \alpha}{\partial x}\right)_{(x,w)=(0,0)}\qquad L=\left(\frac{\partial \alpha}{\partial w}\right)_{(x,w)=(0,0)}\qquad S=\left(\frac{\partial s}{\partial w}\right)_{x=0}.$$

Equation (10) shows how requirement ($\mathrm{S_{FI}}$) is fulfilled if and only of the Jacobian matrix at $x=0$ of system (8), i.e. matrix

$$J_{\mathrm{FI}}=A+BK\,,$$

has all its eigenvalues in the left-half plane.

On the other hand, the closed-loop system in the case of error feedback (7) can be rewritten as

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$$\begin{aligned} \dot{x} &= Ax + BH\xi + Pw + \phi(x,\xi,w) \\ \dot{\xi} &= F\xi + GCx + GQw + \chi(x,\xi,w) \\ \dot{w} &= Sw + \psi(w)\,, \end{aligned} \qquad (11)$$

where $\phi(x,\xi,w)$ and $\psi(w)$ represent higher order terms which vanish at the origin along with their first-order derivatives, A, B, P are the matrices introduced above, while C, Q, F, H, G are matrices defined as follows

$$C = \left(\frac{\partial h}{\partial x}\right)_{(x,w)=(0,0)} \qquad Q = \left(\frac{\partial h}{\partial w}\right)_{(x,w)=(0,0)}$$

$$F = \left(\frac{\partial \eta}{\partial \xi}\right)_{(\xi,e)=(0,0)} \qquad G = \left(\frac{\partial \theta}{\partial e}\right)_{(\xi,e)=(0,0)} \qquad H = \left(\frac{\partial \theta}{\partial \xi}\right)_{\xi=0}.$$

Even in this case it is immediate to realize that requirement (S_{EF}) is fulfilled if and only of the Jacobian matrix at $(x,\xi) = (0,0)$ of system (9), i.e. matrix

$$J_{EF} = \begin{pmatrix} A & BH \\ GC & F \end{pmatrix},$$

has all its eigenvalues in the left-half plane.

From the theory of linear systems it is then easily concluded that requirement (S_{FI}) can be achieved *only if* the pair of matrices (A, B) is stabilizable (i.e. there exists a matrix K such that $A + BK$ has all its eigenvalues in the left-half plane), while requirement (S_{EF}) can be achieved *only if* the pair of matrices (C, A) is detectable (i.e. there exists a matrix G such that $A + GC$ has all its eigenvalues in the left-half plane). These properties of the linear approximation of the process (1), (3) at $(x,u,w) = (0,0,0)$ are indeed necessary conditions for the solvability of the problem of output regulation.

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Biographical Sketches

Alberto Isidori was born in Rapallo, Italy, in 1942. His research interests are primarily focused on mathematical control theory and control engineering. He graduated in electrical engineering from the University of Rome in 1965. Since 1975, he has been Professor of Automatic Control in this University. Since 1989, he also holds a part-time position of Professor of Systems Scie. and Math. at Washington University in St. Louis. He has held visiting positions at various academic/research institutions which include: University of Florida, Gainesville (November 1974), Washington University, St.Louis (August-October 1980, August-December 1983), University of California, Davis (July-August 1983), Arizona State University, Tempe (August-December 1986, April-May 1989), University of Illinois, Urbana (April-May 1987), CINVESTAV, Mexico City (September 1987), University of California, Berkeley (January 1988), CNRS, Paris (May 1988), ETH, Zurich (April-May 1991), Universite Paris-Dauphine, Paris (May 1992), NASA, Langley (November 1996, February 1997).

He is the author of several books: Teoria dei Sistemi (in Italian), with A.Ruberti, 1979; Sistemi di Controllo (in Italian), 1979 and 1992; Nonlinear Control Systems (Springer Verlag), 1985, 1989 and 1995; Topics in Control Theory (Birkhauser), with H.Knobloch and D.Flockerzi, 1993; Output Regulation of Uncertain Nonlinear Systems (Birkhauser), with C.I. Byrnes and F. Delli Priscoli, 1997. He is also editor/coeditor of nine volumes of Conference proceedings and author of over 130 articles, for a large part on the subject of nonlinear feedback design.

He received the G.S.Axelby Outstanding Paper Award from the Control Systems Society of IEEE in 1981, for his technical contributions to the application of differential geometry to the problem of noninteracting control of nonlinear systems, and in 1990, for his technical contributions to the solution of the problem of asymptotic regulation and tracking in nonlinear systems. He also received from the IFAC the Automatica Prize in 1991 for his technical contributions to the application the notion of zero

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dynamics in problems of feedback stabilization. In 1987 he was elected Fellow member of the IEEE "for fundamental contributions to nonlinear control theory".

In 1996, at the opening of 13th IFAC World Congress in San Francisco, Dr. Isidori received the "Giorgio Quazza Medal". This medal is the highest technical award given by the International Federation of Automatic Control, and is presented once every third year for lifetime contributions to automatic control science and engineering. The "Giorgio Quazza Medal" was awarded to Dr. Isidori for "pioneering and fundamental contributions to the theory of nonlinear feedback control".

He has organized or co-organized several international Conferences on the subject feedback design for nonlinear systems. In particular, he was the initiator a permanent series of IFAC Symposia on this topic. He is presently serving in numerous Editorial Boards of major archival journals, which include Automatica, IEEE Transactions on Automatic Control, International Journal of Control, Journal of Mathematical Systems Estimation and Control, International Journal of Robust and Nonlinear Control. He has also served in the program committee of several major international Conferences.

He acted as Program director, in the area of Systems and Control, for the Italian Department of Education from 1983 to 1989. From 1993 to 1996 he served in the Council of IFAC.

Claudio De Persis received his Laurea degree summa cum laude in Electrical Engineering and his doctoral degree in Systems Engineering in 1996 and, respectively, 2000 both from Università di Roma "La Sapienza", Rome, Italy. He held visiting positions in the Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX, and in the Department of Mathematics, University of California, Davis, CA in 1998-1999. From November 1999 to June 2001 he has been a Research Associate in the Department of Systems Science and Mathematics, Washington University in St. Louis, MO. Since July 2001 he has been a Postdoctoral Research Associate in the Department of Electrical Engineering, Yale University, New Haven, CT. On November 1, 2002, he took up his new position as Assistant Professor in the Department of Computer and Systems Science "A. Ruberti", Università di Roma "La Sapienza". He has given contributions to the theory of fault detection for nonlinear systems, switched systems and supervisory control with constraints. His current research interests include observation and control with limited information, hybrid systems, monitoring in large-scale systems, complex systems, networks, modern communication, post-genomic biology.

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